Turbulent transport coefficients from spherical dynamo simulations

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Simulations of stellar dynamos in spherical geometry

- success: oscillatory dynamos with equatorward migration of toroidal magnetic fields

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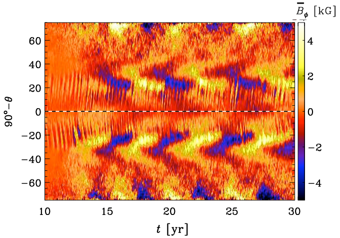

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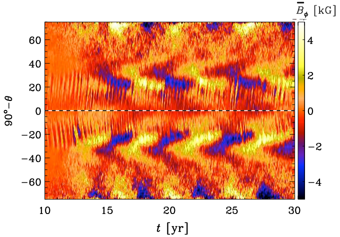

- problems with real stars:
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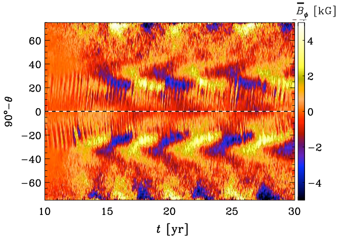

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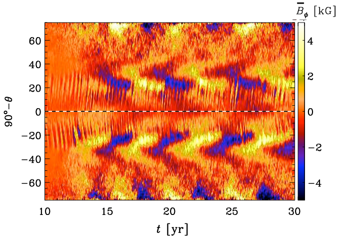

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here simulations considered with:

- spherical wedge geometry: 15° polar cones
- fast rotation: $\Omega = 5\Omega_\odot$
- moderate density contrast: $\lesssim 20$
- various boundary conditions: blackbody vs. fixed T
- various Prandtl numbers: $\frac{\nu}{\eta} = 0.2 \ldots 2.5$, $\frac{\nu}{\chi_{SGS}} = 0.5 \ldots 1$

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deals with *evolution of averaged magnetic field* $\overline{\boldsymbol{B}}$

for spherical bodies: azimuthal average (default)

spherical–harmonic filtering (problematic)

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- descriptive level:
  - qualitative understanding of dynamo
    by identification of crucial effects
  - correlation of specific effects and phenomena
    axisymmetric vs. non-axisymmetric modes
    equatorially symmetric vs. antisymmetric vs. hemispherical
    equatorward vs. poleward migration
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- predictive level:
  - growth rates, eigenfunctions of kinematic modes *(doable)*
  - long-term simulations, producing grand extrema or random polarity reversals by intrinsic nonlinearities *(not doable now)*

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in terms of $\overline{\boldsymbol{B}}$, e.g. by ansatz

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- analytical with approximations – strongly limited
- by *testfield method*

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for given $\boldsymbol{u}, \overline{\boldsymbol{U}}$ and N test fields $\overline{\boldsymbol{B}}^k, \; k = 1, \ldots, N$

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   solution unique, if

   - N chosen appropriately
   - test fields independent

Testfield method for axisymmetric mean fields

$\Longrightarrow$ in spherical coords (r, θ, ϕ):

$$\overline{\boldsymbol{B}} = \overline{B}_r(r,\theta)\mathbf{e}_r + \overline{B}_\theta(r,\theta)\mathbf{e}_\theta + \overline{B}_\phi(r,\theta)\mathbf{e}_\phi$$

$\Longrightarrow$

$$\overline{\mathcal{E}}_\kappa = \widetilde{a}_{\kappa\lambda}\overline{B}_\lambda + \tilde{b}_{\kappa\lambda r}\frac{\partial \overline{B}_\lambda}{\partial r} + \tilde{b}_{\kappa\lambda\theta}\frac{1}{r}\frac{\partial \overline{B}_\lambda}{\partial \theta}\,, \quad \kappa, \lambda = r, \theta, \phi$$

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27 independent coefficients !

“effect-wise"

$$\overline{\mathcal{E}} = \boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} + \boldsymbol{\gamma}\times\overline{\boldsymbol{B}} - \boldsymbol{\beta}\cdot\mathrm{curl}\,\overline{\boldsymbol{B}} - \boldsymbol{\delta}\times\mathrm{curl}\,\overline{\boldsymbol{B}} - \boldsymbol{\kappa}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{sym})}$$

↑ turbulent pumping

↑ turbulent diffusivity

↑ "$\boldsymbol{\Omega}\times\boldsymbol{J}$" effect

$\boldsymbol{\alpha}, \boldsymbol{\beta}$ – symmetric, $\boldsymbol{\kappa}$ – symmetric in 2nd and 3rd indices

Testfields

simplest choice: linear, e.g.

No.	1	2	3	4	5	6	7	8	9
B_r	1	0	0	r	0	0	θ	0	0
B_θ	0	1	0	0	r	0	0	θ	0
B_ϕ	0	0	1	0	0	r	0	0	θ

Schrinner et al. 2007

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Schrinner et al. 2007

- some irregular or not solenoidal
- violate boundary conditions
- yet suitable
- within class of linear functions: result independent of choice

Equatorial symmetries in coefficients

$\exists$ special solutions of full MHD problem:

$$\boldsymbol{U}, \rho, s \quad \text{equatorially symmetric}, \quad \boldsymbol{B} \begin{cases} \text{symmetric} \\ \text{antisymmetric} \end{cases} \quad \Longrightarrow$$

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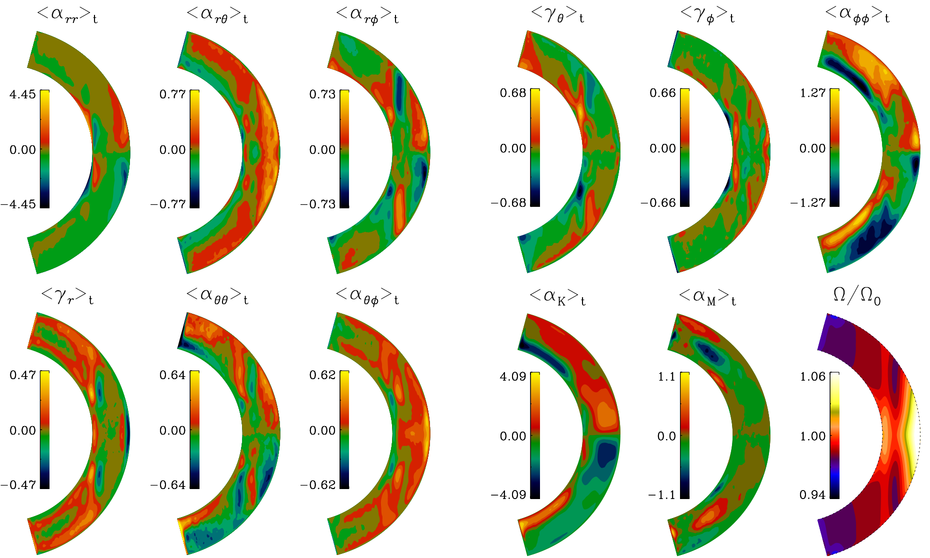
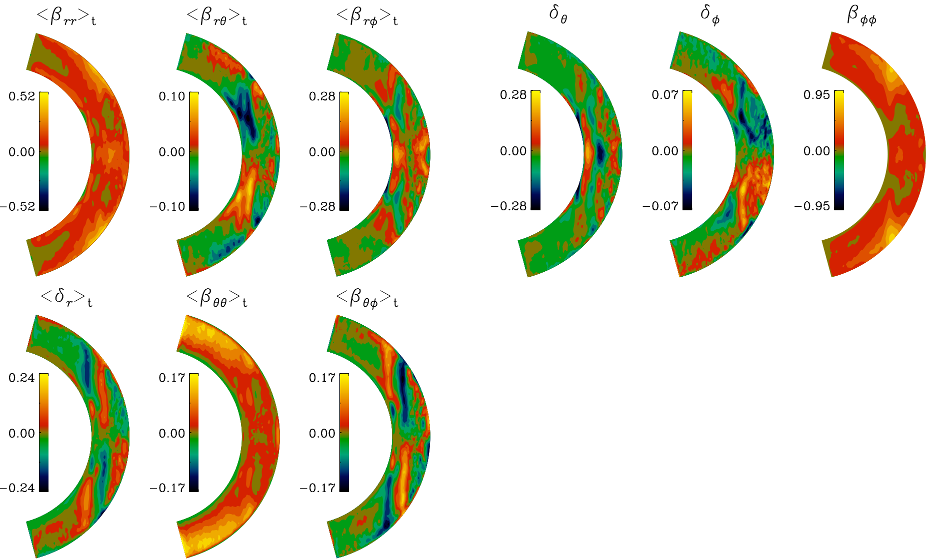
- diagonal components of $\boldsymbol{\alpha}$, $\alpha_{r\phi}$, γ_θ *antisymmetric*
  all other *symmetric*
- diagonal components of $\boldsymbol{\beta}$, $\beta_{r\phi}$, δ_θ *symmetric*
  all other *antisymmetric*

Time-averaged components of α and γ (normalized by $u'_{\rm rms}/3$)



Time-averaged components of β and δ (normalized by $\tau u'^2_{\rm rms}/3$)



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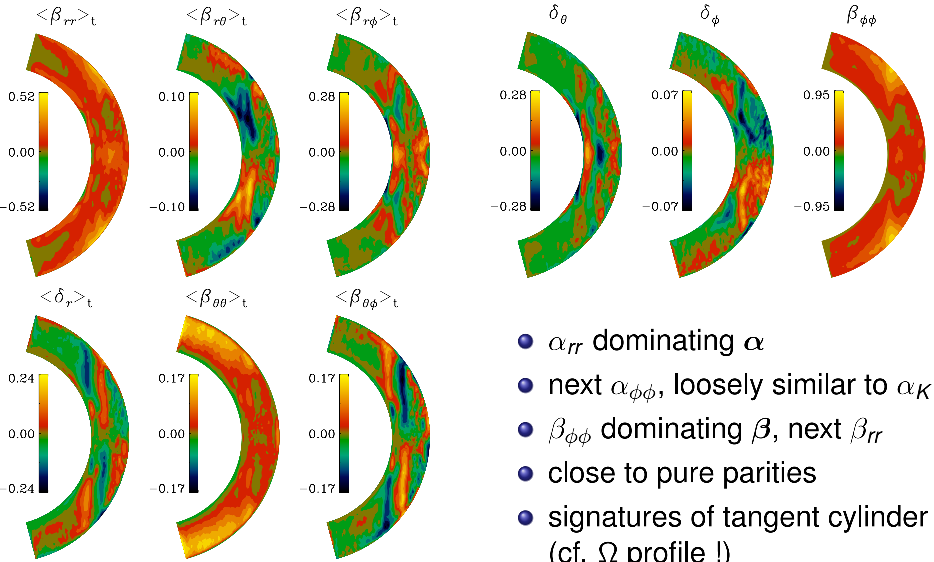
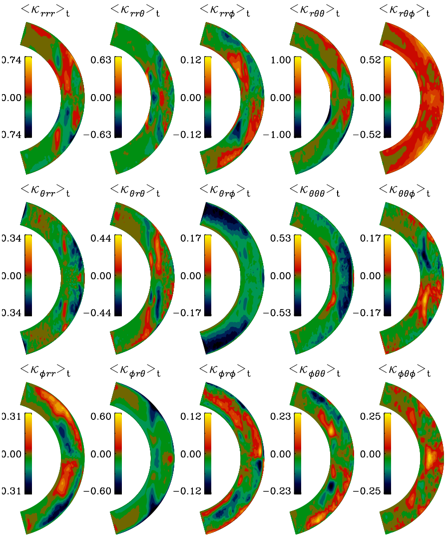


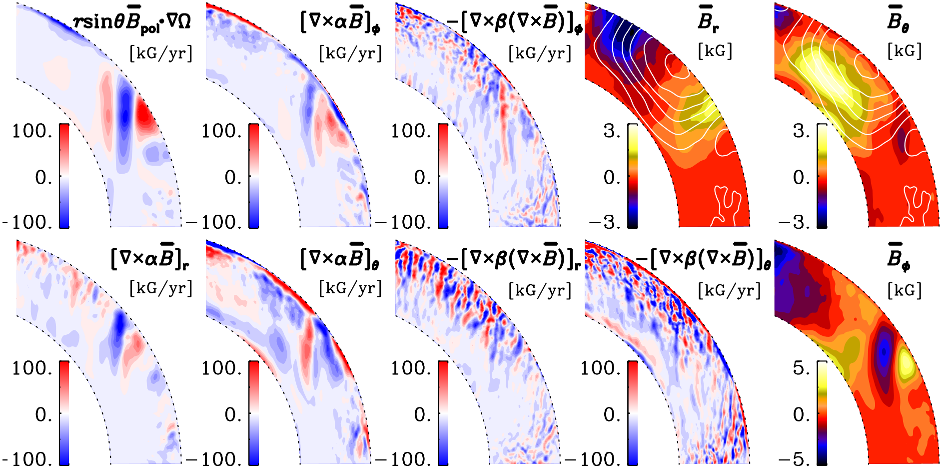
- α_{rr} dominating $\boldsymbol{\alpha}$
- next $\alpha_{\phi\phi}$, loosely similar to α_K
- $\beta_{\phi\phi}$ dominating $\boldsymbol{\beta}$, next β_{rr}
- close to pure parities
- signatures of tangent cylinder (cf. Ω profile !)

Time-averaged components of κ (normalized by $\tau u'^2_{\rm rms}/3$)



Main drivers of $\overline{B}$ evolution

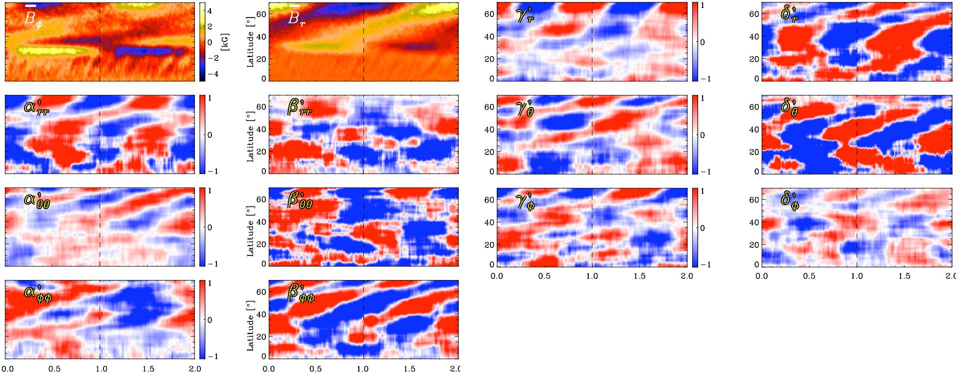
at maximum of “typical cycle”:



white – field lines of poloidal field $\overline{B}_r$, $\overline{B}_\theta$

Cyclic modulation of transport coefficients

over "typical cycle", $\alpha' = \alpha - \langle\alpha\rangle_t$ etc:



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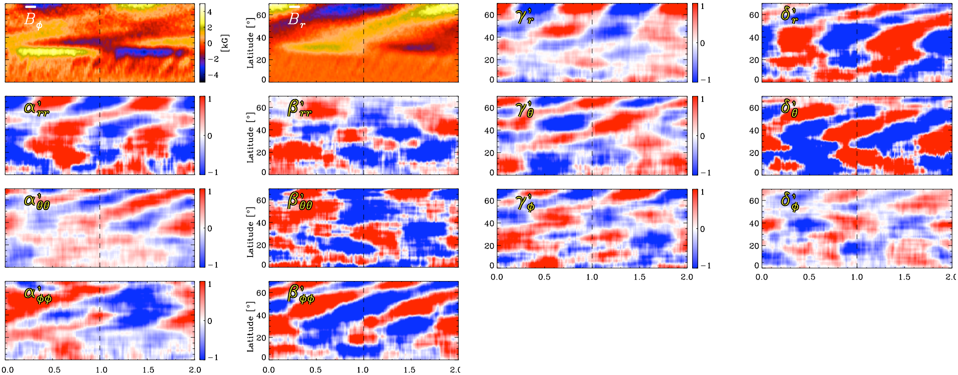

- typical modulation by $2f_{\rm cycle}$
- $\alpha_{\theta\theta,\phi\phi}$, low latitudes: *only by* $f_{\rm cycle}$
- conflict with *quadratic* Lorentz force?

Signature of small-scale dynamo action?

consider *primary magnetic turbulence*, i.e. $\boldsymbol{b} \not\to \mathbf{0}$ for $\overline{\boldsymbol{B}} \equiv \mathbf{0}$
= *small-scale dynamo* $(\boldsymbol{u}_0, \boldsymbol{b}_0)$

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consider lowest order of $\overline{\boldsymbol{B}}$ in transport coefficients

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$\longrightarrow \overline{\boldsymbol{u}_0 \times \boldsymbol{b}_{\overline{\boldsymbol{B}}}}$ quadratic in $\overline{\boldsymbol{B}} \longrightarrow$ coefficients *linear* in $\overline{\boldsymbol{B}}$

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- farther away:
  - transport coefficients for momentum & heat transport
  - identification of the $\overline{\boldsymbol{B}}$ dependence $\Longrightarrow$ predictive models