Answer Set Programming as SAT modulo Acyclicity

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Answer Set Programming

Answer set programming (ASP) features a rule-based syntax subject to answer-set semantics.

Problem $\xrightarrow{\text{Formalize}}$ Program $P$ $\xrightarrow{\text{Ground}}$ $\text{Gnd}(P)$ $\xrightarrow{\text{Search}}$ $\text{SM}(P)$ $\xrightarrow{\text{Extract}}$ Solution(s)

Some native answer set solvers:

- CLASP  http://potassco.sourceforge.net/
- CMODELS  http://www.cs.utexas.edu/~tag/cmodels/
- DLV  http://www.dlvsystem.com/
- SMODELS  http://research.ics.aalto.fi/software/
Example: SuDoku Puzzle

number(1..9).
border(1). border(4). border(7).
region(X,Y) :- border(X), border(Y).

\[
\begin{array}{ccc}
1 & 9 & 3 \\
4 & 6 & 8 \\
7 & 5 & 2 \\
\end{array} \quad \begin{array}{ccc}
8 & 6 & 7 \\
5 & 3 & 2 \\
1 & 4 & 9 \\
\end{array} \quad \begin{array}{ccc}
4 & 2 & 5 \\
9 & 1 & 7 \\
6 & 8 & 3 \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & 1 & 6 \\
5 & 3 & 4 \\
9 & 8 & 7 \\
\end{array} \quad \begin{array}{ccc}
3 & 9 & 5 \\
4 & 7 & 3 \\
9 & 1 & 8 \\
\end{array} \quad \begin{array}{ccc}
5 & 9 & 8 \\
7 & 6 & 2 \\
3 & 4 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
2 & 1 & 6 \\
8 & 7 & 5 \\
3 & 4 & 9 \\
\end{array} \quad \begin{array}{ccc}
7 & 8 & 1 \\
6 & 2 & 4 \\
1 & 3 & 9 \\
\end{array} \quad \begin{array}{ccc}
2 & 5 & 6 \\
\end{array}
\]

1 \{ value(X,Y,N):number(X):number(Y):
\quad X1<=X: X<=X1+2: Y1<=Y: Y<=Y1+2 \} 1
\quad :- number(N), region(X1,Y1).

:- 2 \{value(X,Y,N):number(N)}}, number(X), number(Y).
:- 2 \{value(X,Y,N):number(Y)}}, number(N), number(X).
:- 2 \{value(X,Y,N):number(X)}}, number(N), number(Y).
Example: Running the Solver

$ gringo sudoku.lp royle.lp | clasp 0
clasp version 3.0.4
Reading from stdin
Solving...
Answer: 1
value(1,3,2) value(1,9,1) value(2,2,7) value(2,5,3) value(3,5,4)
value(3,7,2) value(4,4,2) value(5,7,4) value(5,8,3) value(6,1,1)
value(6,3,5) value(6,4,6) value(7,5,7) value(8,2,3) value(9,4,1)
value(9,9,5) value(1,2,9) value(3,1,8) ...
Answer: 2
value(1,3,2) value(1,9,1) value(2,2,7) value(2,5,3) value(3,5,4)
value(3,7,2) value(4,4,2) value(5,7,4) value(5,8,3) value(6,1,1)
value(6,3,5) value(6,4,6) value(7,5,7) value(8,2,3) value(9,4,1)
value(9,9,5) value(3,1,9) ...
SATISFIABLE
Models : 2
...

Aalto University
School of Science
Key Features of ASP

- Typical ASP encodings follow a three-phase **design**:
  1. Generate the solution candidates
  2. Define the required concepts
  3. Test if a candidate satisfies its criteria

- **Default negation** favors concise encodings.

- **Basic database operations** are definable in terms of rules:
  - Projection: node($X$) ← edge($Y$, $X$).
  - Union: node($X$) ← edge($Y$, $X$). node($Y$) ← edge($Y$, $X$).
  - Intersection: symm($X$, $Y$) ← edge($X$, $Y$), edge($Y$, $X$).
  - Complement: unidir($X$, $Y$) ← edge($X$, $Y$), not edge($Y$, $X$).

- **Recursive definitions** are also supported:
  path($X$, $Y$) ← edge($X$, $Z$), path($Z$, $Y$), node($Y$).
Translation-Based ASP

ASP can be implemented by translating ground programs into:

- **Boolean Satisfiability** (SAT)
  
  [J., ECAI, 2004; J. and Niemelä, MG-65, 2010]

- **Integer Difference Logic** (IDL)
  
  [Niemelä, AMAI, 2008; J. et al., LPNMR, 2009]

- **Integer Programming** (IP)
  
  [Liu et al., KR, 2012]

- **Bit-Vector Logic** (BV)
  
  [Nguyen et al., INAP, 2011; Extended in 2013]

☞ Existing solver technology can be harnessed for ASP!
Motivation

- Complexities of translations vary in program length $n$:
  - $O(n)$: IDL, IP, BV
  - $O(n \times \log_2 n)$: SAT [J., ECAI 2004]
  - $O(n^2)$: SAT [Lin & Zhao, IJCAI 2003]
  - $O(2^n)$: SAT [Lin & Zhao, AIJ 2004]

- What would be a minimal extension of SAT such that
  1. a linear embedding from ASP is enabled and
  2. the extension is efficiently implementable?

- In this paper, we consider embeddings into an extension based on graphs subject to an acyclicity constraint:

  M. Gebser, T. Janhunen, and J. Rintanen:
  “Satisfiability Modulo Graphs: Acyclicity” [JELIA 2014].
Outline

Formalisms of Interest

Translating Programs into SAT modulo Acyclicity

Implementation and Experiments

Conclusion
Source Formalism: Normal Programs

- Normal logic programs (NLPs) consist of rules of the form:
  \[ a \leftarrow b_1, \ldots, b_n, \text{not } c_1, \ldots, \text{not } c_m. \]

- The semantics is given by stable models, also known as answer sets, satisfying [Gelfond and Lifschitz, ICLP, 1988]:
  \[ M = LM(P^M). \]

Example
Consider the following program:
\[ a \leftarrow b. \quad a \leftarrow c. \quad b \leftarrow a. \quad c \leftarrow \text{not } d. \quad d \leftarrow \text{not } c. \]
\[ \implies M_1 = \{a, b, c\} \text{ is stable but } M_2 = \{a, b, d\} \text{ is not.} \]
Target Formalism: Syntax

A theory in SAT modulo acyclicity (ACYC) is a tuple \( \langle X, C, N, A, I \rangle \) where

1. \( C \) is a set of \textit{clauses} based on propositional variables in \( X \),
2. \( G = \langle N, A \rangle \) is a \textit{directed graph} with a finite set of nodes \( N \) and arcs \( A \subseteq N \times N \), and
3. \( I : A \rightarrow X \) is a \textit{labeling} that assigns a propositional variable \( l(u, v) \) to every arc \( \langle u, v \rangle \in A \) in the graph \( G \).

Example

Rewriting our NLP using \( N = \{a, b\} \) and \( E = \{\langle a, b \rangle, \langle b, a \rangle\} \):

\[
\begin{align*}
a \lor \neg b, & \quad a \lor \neg c, & \quad \neg a \lor b \lor c, & \quad b \lor \neg a, & \quad \neg b \lor a, \\
c \lor d, & \quad \neg c \lor \neg d, & \quad \neg a \lor c \lor e_{\langle a, b \rangle}, & \quad \neg b \lor e_{\langle b, a \rangle}.
\end{align*}
\]
Target Formalism: Semantics

An ACYC theory $T = \langle X, C, N, A, l \rangle$ is satisfied by an interpretation $M \subseteq X$, denoted $M \models T$, iff

1. $M \models C$ and
2. $\langle N, A_M \rangle$ with $A_M = \{ \langle u, v \rangle \in A \mid M \models l(u, v) \}$ is acyclic.

Example

Recall the theory $T$ from our running example:

$$a \lor \neg b, \quad a \lor \neg c, \quad \neg a \lor b \lor c, \quad b \lor \neg a, \quad \neg b \lor a,$$

$$c \lor d, \quad \neg c \lor \neg d, \quad \neg a \lor c \lor e_{\langle a, b \rangle}, \quad \neg b \lor e_{\langle b, a \rangle}.$$ 

$\implies$ $M_1 = \{ a, b, c, e_{\langle b, a \rangle} \} \models T$ but $M_2 = \{ a, b, d, e_{\langle a, b \rangle}, e_{\langle b, a \rangle} \} \not\models T.$
Applications in Sight

Acyclicity constraints lend themselves for many purposes:

- Specifying a variety of topological structures:
  - Trees and forests (both directed and undirected)
  - Directed acyclic graphs (DAGs)
  - Chordal graphs

- Hamiltonian cycles

- Formalizing paths and reachability in general
General Translation from ASP to ACYC

- The classical models of the completion $\text{Comp}(P)$ coincide with the supported models $P$ [Apt et al., 1988].
- The strong groundedness of stable models can be captured by assigning numbers/ordinals to atoms [Elkan, AIJ 1990; Fages, JMLCS 1994; Erdem & Lifschitz, TPLP 2003].
- We follow the linear translation into IDL based on level rankings [Niemelä, AMAI 2008; J. et al., LPNMR 2009].
- The translation has to be applied only to atoms $a \in \text{At}(P)$ having a non-trivial component $\text{SCC}(a)$ with $|\text{SCC}(a)| > 1$.

In our running example, we have $\text{SCC}(a) = \{a, b\} = \text{SCC}(b)$:

\[
\begin{align*}
a & \leftarrow b. \\
a & \leftarrow c. \\
b & \leftarrow a. \\
c & \leftarrow \text{not } d. \\
d & \leftarrow \text{not } c.
\end{align*}
\]
Identifying Rule Bodies

Following [Tseitin, 1968], the body $B(r)$ of a defining rule $r \in \text{Def}_P(a)$ is given a new name $bd_r$ by

1. the clause $bd_r \lor \bigvee_{b \in B^+(r)} \neg b \lor \bigvee_{c \in B^-(r)} c$,
2. for each $b \in B^+(r)$, the clause $\neg bd_r \lor b$, and
3. for each $c \in B^-(r)$, the clause $\neg bd_r \lor \neg c$.

Effectively, we have $bd_r \leftrightarrow \bigwedge_{b \in B^+(r)} b \land \bigwedge_{c \in B^-(r)} \neg c$.

Example

<table>
<thead>
<tr>
<th>Rule:</th>
<th>Translation:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \leftarrow b$.</td>
<td>$bd_1 \lor \neg b$ $\neg bd_1 \lor b$</td>
</tr>
<tr>
<td>$a \leftarrow c$.</td>
<td>$bd_2 \lor \neg c$ $\neg bd_2 \lor c$</td>
</tr>
<tr>
<td>$b \leftarrow a$.</td>
<td>$bd_3 \lor \neg a$ $\neg bd_3 \lor a$</td>
</tr>
</tbody>
</table>
Well Support from Internal Rules

For the well-support provided by a rule $r \in \text{IDef}_P(a)$:

1. The clause $\text{ws}_r \lor \neg \text{bd}_r \lor \bigvee_{b \in B^+(r) \cap \text{SCC}(a)} \neg e_{\langle a,b \rangle}$.
2. The clause $\neg \text{ws}_r \lor \text{bd}_r$.
3. For each $b \in B^+(r) \cap \text{SCC}(a)$, the clause $\neg \text{ws}_r \lor e_{\langle a,b \rangle}$.

$\implies$ Effectively, we have $\text{ws}_r \leftrightarrow \text{bd}_r \land \bigwedge_{b \in B^+(r) \cap \text{SCC}(a)} e_{\langle a,b \rangle}$.

Example

<table>
<thead>
<tr>
<th>Internal rule:</th>
<th>$a \leftarrow b.$</th>
<th>$b \leftarrow a.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translation:</td>
<td>$\text{ws}_1 \lor \neg \text{bd}<em>1 \lor \neg e</em>{\langle a,b \rangle}$</td>
<td>$\text{ws}_3 \lor \neg \text{bd}<em>3 \lor \neg e</em>{\langle b,a \rangle}$</td>
</tr>
<tr>
<td></td>
<td>$\neg \text{ws}_1 \lor \text{bd}_1$</td>
<td>$\neg \text{ws}_3 \lor \text{bd}_3$</td>
</tr>
<tr>
<td></td>
<td>$\neg \text{ws}<em>1 \lor e</em>{\langle a,b \rangle}$</td>
<td>$\neg \text{ws}<em>3 \lor e</em>{\langle b,a \rangle}$</td>
</tr>
</tbody>
</table>
Enforcing Support for Atoms

For the definition $\text{Def}_P(a)$ of an atom $a$ in a program $P$:

1. For each $r \in \text{Def}_P(a)$, the clause $a \lor \neg \text{bd}_r$.
2. The clause $\neg a \lor \bigvee_{r \in E\text{Def}_P(a)} \text{bd}_r \lor \bigvee_{r \in I\text{Def}_P(a)} \text{ws}_r$.

$\implies$ Effectively, this entails that $a \leftrightarrow \bigvee_{r \in \text{Def}_P(a)} B(r)$.

Example

<table>
<thead>
<tr>
<th>Definition:</th>
<th>$a \leftarrow b$</th>
<th>$a \leftarrow c$</th>
<th>$b \leftarrow a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translation:</td>
<td>$a \lor \neg \text{bd}_1$, $a \lor \neg \text{bd}_2$</td>
<td>$\neg a \lor \text{ws}_1 \lor \text{bd}_2$</td>
<td>$b \lor \neg \text{bd}_3$</td>
</tr>
</tbody>
</table>
Overall Properties of the Translation

- The resulting translation $\text{Tr}_{\text{ACYC}}(P)$ of a normal program $P$ is linear in the length of $P$.
- A one-to-many correspondence between the stable models of $P$ and the models of $\text{Tr}_{\text{ACYC}}(P)$ is obtained.

Proposition

Let $P$ be a normal logic program and $\text{Tr}_{\text{ACYC}}(P)$ its translation into SAT modulo acyclicity.

1. If $M \in \text{SM}(P)$, then there is a model $N \models \text{Tr}_{\text{ACYC}}(P)$ such that $M = N \cap \text{At}(P)$.
2. If $N \models \text{Tr}_{\text{ACYC}}(P)$, then $M \in \text{SM}(P)$ for $M = N \cap \text{At}(P)$.
Extension: Disabling Edges Dynamically

- An edge variable $e_{\langle a, b \rangle}$ can be falsified if
  1. $a$ is known to be false,
  2. $a$ has an externally supporting rule, or
  3. $a$ has an internally supporting rule $r \in IDef_P(a)$ such that $b \not\in B^+(r)$.

- The extended translation $\text{Tr}^+_\text{ACYC}(P)$ gives rise to a similar but tighter correspondence of models.

Example

<table>
<thead>
<tr>
<th>Definition:</th>
<th>Case 1:</th>
<th>Case 2:</th>
<th>Case 3:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \leftarrow b$. $a \leftarrow c$.</td>
<td>$b \leftarrow a$.</td>
<td>$b \leftarrow a$.</td>
<td>$b \leftarrow a$.</td>
</tr>
<tr>
<td>$a \lor \neg e_{\langle a, b \rangle}$</td>
<td>$b \lor \neg e_{\langle b, a \rangle}$</td>
<td>$b \lor \neg e_{\langle b, a \rangle}$</td>
<td>$b \lor \neg e_{\langle b, a \rangle}$</td>
</tr>
<tr>
<td>$\neg bd_2 \lor \neg e_{\langle a, b \rangle}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>
Implementation

- For tool interoperability, the SMODELS format is used as an intermediate format for representing ground programs.
- Extended rules, such as choice, cardinality, and weight rules may have to be translated away using LP2NORMAL2.
- To enable cross-translation for different back-end solvers,
  1. the input program is instrumented with auxiliary atoms and auxiliary rules corresponding to $\text{Tr}^{+}_{\text{ACYC}}$ and
  2. the completion is produced in the target format of interest.
- Our tools produce a number of output formats:
  1. DIMACS with optional ACYC and MAXSAT extensions
  2. SMT Library 2.0 (QF_IDL and QF_BV fragments)
  3. PB format
  4. CPLEX
# Tool Support

<table>
<thead>
<tr>
<th>Tool</th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>gringo / lparse</td>
<td></td>
</tr>
<tr>
<td>lpstrip</td>
<td></td>
</tr>
<tr>
<td>lpcat</td>
<td></td>
</tr>
<tr>
<td>lp2normal2</td>
<td>--</td>
</tr>
<tr>
<td>lp2acyc</td>
<td></td>
</tr>
<tr>
<td>lp2sat [-g]</td>
<td>acyc2solver [--diff]</td>
</tr>
<tr>
<td></td>
<td>[--bv]</td>
</tr>
<tr>
<td></td>
<td>[--pb]</td>
</tr>
<tr>
<td></td>
<td>[--mip]</td>
</tr>
</tbody>
</table>

The tool collection is published under:

http://research.ics.aalto.fi/software/asp/lp2acyc/
## Experiments

<table>
<thead>
<tr>
<th>Problem Size</th>
<th>Hamilton</th>
<th></th>
<th>Tree</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>150</td>
<td>25</td>
<td>50</td>
<td>75</td>
<td>100</td>
</tr>
<tr>
<td>CLASP</td>
<td>0.95</td>
<td>20.16</td>
<td>4.37</td>
<td>1193.09</td>
<td>1495.32</td>
</tr>
<tr>
<td>ACYCGLUCOSE</td>
<td>0.07</td>
<td>0.15</td>
<td>0.74</td>
<td>315.83</td>
<td>999.07</td>
</tr>
<tr>
<td>ACYCMINISAT</td>
<td>0.04</td>
<td>0.12</td>
<td>0.83</td>
<td>544.43</td>
<td>1025.02</td>
</tr>
<tr>
<td>Z3</td>
<td>2.45</td>
<td>50.64</td>
<td>4.75</td>
<td>1208.36</td>
<td>1726.56</td>
</tr>
<tr>
<td>ACYCGLUCOSE-\text{Tr}_{\text{ACYC}}</td>
<td>0.93</td>
<td>13.75</td>
<td>1.40</td>
<td>271.93</td>
<td>973.22</td>
</tr>
<tr>
<td>ACYCMINISAT-\text{Tr}_{\text{ACYC}}</td>
<td>0.76</td>
<td>7.28</td>
<td>0.80</td>
<td>484.92</td>
<td>879.18</td>
</tr>
<tr>
<td>Z3-\text{Tr}_{\text{ACYC}}</td>
<td>35.80</td>
<td>331.11</td>
<td>6.30</td>
<td>1178.44</td>
<td>2266.66</td>
</tr>
<tr>
<td>ACYCGLUCOSE-\text{Tr}^{+}_{\text{ACYC}}</td>
<td>0.04</td>
<td>0.18</td>
<td>1.09</td>
<td>264.28</td>
<td>931.28</td>
</tr>
<tr>
<td>ACYCMINISAT-\text{Tr}^{+}_{\text{ACYC}}</td>
<td>0.08</td>
<td>0.32</td>
<td>0.77</td>
<td>473.64</td>
<td>852.78</td>
</tr>
<tr>
<td>Z3-\text{Tr}^{+}_{\text{ACYC}}</td>
<td>27.72</td>
<td>239.83</td>
<td>7.03</td>
<td>1230.51</td>
<td>1976.20</td>
</tr>
</tbody>
</table>
ASP Competition 2014

The LP2GRAPH system was based on the translation $T_{ACYC}^{+}$ and using ACYCGlucose as the back-end solver.

[https://www.mat.unical.it/aspcomp2014/]
Conclusion

- Translation-based ASP aims to exploit
  - the expressive power of ASP and
  - the potential behind existing solver technology.

- The translation from ASP into SAT modulo acyclicity
  - is linear and
  - preserves stable models up to original signature.

- The cross-translation of ASP is enabled by
  - a suitable intermediate format and
  - postponing format-specific aspects to the last step.

- Future extensions:
  - Support for further formats and solver types
  - Covering optimization more widely