A primer to Strassen semirings and their asymptotic spectra

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Summary

- This talk is a short primer to **Strassen semirings** and their **asymptotic spectra** based on a recent exposition by Wigderson and Zuiddam https://staff.fnwi.uva.nl/j.zuiddam/papers/convexity.pdf
- We aim to look at and motivate three examples:
  1. $\mathbb{R}_{\geq 1}$
  2. Restriction-preordered **tensors** [asymptotic tensor rank]
  3. Cohomomorphism-preordered **graphs** [Shannon capacity]
- Time permitting, we will also seek to sketch Strassen's **duality** between **asymptotic rank** and the **asymptotic spectrum** formed by the **monotone homomorphisms**

A short boot camp of definitions

- [Commutative] semiring
- Partial order, total order, preorder
- Semiring preorder
- Strassen preorder
- Strassen-preordered commutative semiring [Strassen semiring]

Commutative semiring

- A three-tuple $(\mathcal{R}, +, \cdot)$ consisting of a set $\mathcal{R}$ and two binary operations $+ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ and $\cdot : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ is called a **commutative semiring** if
  1. for all $a, b, c \in \mathcal{R}$ we have $(a + b) + c = a + (b + c)$, [associativity of $+$]
  2. for all $a, b \in \mathcal{R}$ we have $a + b = b + a$, [commutativity of $+$]
  3. for all $a, b, c \in \mathcal{R}$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, [associativity of $\cdot$]
  4. for all $a, b \in \mathcal{R}$ we have $a \cdot b = b \cdot a$, [commutativity of $\cdot$]
  5. for all $a, b, c \in \mathcal{R}$ we have $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, and [distributivity of $\cdot$ over $+$]
  6. there exists a $1 \in \mathcal{R}$ such that for all $a \in \mathcal{R}$ we have $1 \cdot a = a$ [multiplicative unit]

- Example: $\mathbb{R}_{\geq 1}$ (the set of real numbers at least 1)

Binary relations

- A binary relation $\leq$ on a set $\mathcal{R}$ is
  1. **reflexive** if for all $a \in \mathcal{R}$ we have $a \leq a$
  2. **symmetric** if for all $a, b \in \mathcal{R}$ we have $a \leq b$ implies $b \leq a$
  3. **antisymmetric** if for all $a, b \in \mathcal{R}$ we have $a \leq b$ and $b \leq a$ imply $a = b$
  4. **transitive** if for all $a, b, c \in \mathcal{R}$ we have $a \leq b$ and $b \leq c$ imply $a \leq c$
  5. **total** if for all $a, b \in \mathcal{R}$ at least one of $a \leq b$ or $b \leq a$ holds
  6. **equivalence** if it is reflexive, symmetric, and transitive
  7. **partial order** if it is reflexive, antisymmetric, and transitive
  8. **preorder** if it is reflexive and transitive
  9. **order** if it is a partial order and total
- Let P, Q be binary relations on a set $\mathcal{R}$; we say that Q **extends** P if $P \subseteq Q$

Strassen preorder

- Let $\mathcal{R}$ be a commutative semiring and let $\mathbb{N} = \{0, 1, 2, \ldots\}$
- A preorder $\leq$ on $\mathcal{R}$ is a **semiring preorder** if for every $a, b, c, d \in \mathcal{R}$ we have that $a \leq b$ and $c \leq d$ imply $a + c \leq b + d$ and $ac \leq bd$
- A semiring preorder $\leq$ on $\mathcal{R}$ is a **Strassen preorder** if
  1. for every $m, n \in \mathbb{N}_{\geq 1}$ we have $m \leq_{\mathbb{N}} n$ if and only if $m_{\mathcal{R}} \leq_{\mathcal{R}} n_{\mathcal{R}}$, [embedding of the natural numbers]
  2. for every $a, b \in \mathcal{R}$ it holds that $a \leq a + b$, and [sum rule]
  3. for every $a \in \mathcal{R}$ it holds that $1_{\mathcal{R}} \leq a \leq n_{\mathcal{R}}$ for some $n \in \mathbb{N}_{\geq 1}$ [strong Archimedean property]
- The pair $(\mathcal{R}, \leq)$ with $\leq$ a Strassen preorder on $\mathcal{R}$ is a **Strassen semiring**
- *Example.*
  $(\mathbb{R}_{\geq 1}, \leq_{\mathbb{R}})$ is a Strassen semiring

Rank and subrank

- Let $(\mathcal{R}, \leq)$ be a Strassen semiring
- For $a \in \mathcal{R}$, the **rank** $R(a)$ is the least number $n \in \mathbb{N}_{\geq 1}$ such that $a \leq n_{\mathcal{R}}$
- For $a \in \mathcal{R}$, the **subrank** $Q(a)$ is the greatest number $m \in \mathbb{N}_{\geq 1}$ such that $m_{\mathcal{R}} \leq a$
- *Remarks.* Rank is sub-additive, sub-multiplicative, normalized, and $\leq$-monotone; subrank is super-additive, super-multiplicative, normalized, and $\leq$-monotone
- *Example.*
  For $(\mathbb{R}_{\geq 1}, \leq_{\mathbb{R}})$, we have $R(a) = \lceil a \rceil$ and $Q(a) = \lfloor a \rfloor$

Asymptotic rank and asymptotic subrank

- Let $(\mathcal{R}, \leq)$ be a Strassen semiring
- For $a \in \mathcal{R}$, the **asymptotic rank** $\tilde{R}(a)$ is the infimum of $\{R(a^n)^{1/n} : n \in \mathbb{N}_{\geq 1}\}$
- For $a \in \mathcal{R}$, the **asymptotic subrank** $\tilde{Q}(a)$ is the supremum of $\{Q(a^n)^{1/n} : n \in \mathbb{N}_{\geq 1}\}$
- *Remarks.*
    1. Asymptotic rank is sub-additive, sub-multiplicative, normalized, and $\leq$-monotone
    2. Asymptotic subrank is super-additive, super-multiplicative, normalized, and $\leq$-monotone
    3. By Fekete's lemma, the infimum and supremum can be replaced by limits $n \to \infty$
- *Example.*
  For $(\mathbb{R}_{\geq 1}, \leq_{\mathbb{R}})$, we have $\tilde{R}(a) = \tilde{Q}(a) = a$

Example: Tensors (1/2)

- Let $\mathbb{F}$ be a field and let $k \in \mathbb{N}_{\geq 1}$; let us abbreviate $[k] = \{0, 1, \ldots, k-1\}$
- For $d_1, d_2, \ldots, d_k \in \mathbb{N}_{\geq 1}$, an array $T \in \mathbb{F}^{d_1 \times d_2 \times \cdots \times d_k}$ is k-**tensor** over $\mathbb{F}$ of **shape** $d_1 \times d_2 \times \cdots \times d_k$
- A tensor T is *nonzero* if there exist $i_1 \in [d_1]$, $i_2 \in [d_2]$, ..., $i_k \in [d_k]$ such that $T_{i_1, i_2, \ldots, i_k} \neq 0$
- *Example.* Matrices are 2-tensors
- We will fix $k = 3$ in what follows and abbreviate "3-tensor" to "tensor"
- Two tensors are **equivalent** if they have the same shape and one can be obtained by the other by rearranging the slices, the rows, and the columns of the tensor
- Let $\mathcal{T}$ be the set of all equivalence classes of *nonzero* tensors (of any shape)

Example: Tensors (2/2)

- Let $T \in \mathbb{F}^{d_1 \times d_2 \times d_3}$ and $T' \in \mathbb{F}^{d'_1 \times d'_2 \times d'_3}$ be tensors
- The **direct sum** $T \oplus T' \in \mathbb{F}^{(d_1+d'_1)\times(d_2+d'_2)\times(d_3+d'_3)}$ is defined for all $j_1 \in [d_1 + d'_1]$, $j_2 \in [d_2 + d'_2]$, $j_3 \in [d_3 + d'_3]$ by

$$(T \oplus T')_{j_1,j_2,j_3} = \begin{cases} T_{j_1,j_2,j_3} & \text{if } j_1 < d_1,\, j_2 < d_2,\, j_3 < d_3; \\ T'_{j_1-d_1,j_2-d_2,j_3-d_3} & \text{if } j_1 \geq d_1,\, j_2 \geq d_2,\, j_3 \geq d_3; \\ 0 & \text{otherwise} \end{cases}$$

- The **Kronecker product** $T \otimes T' \in \mathbb{F}^{(d_1 d'_1)\times(d_2 d'_2)\times(d_3 d'_3)}$ is defined for all $i_1 \in [d_1]$, $i'_1 \in [d'_1]$, $i_2 \in [d_2]$, $i'_2 \in [d'_2]$, $i_3 \in [d_3]$, $i'_3 \in [d'_3]$ by $(T \otimes T')_{i_1 d'_1 + i'_1,\, i_2 d'_2 + i'_2,\, i_3 d'_3 + i'_3}$
- The direct sum and the Kronecker product are well-defined on $\mathcal{T}$
- Observing that the $1 \times 1 \times 1$ tensor $1_{\mathcal{T}}$ with the entry $1_{\mathbb{F}}$ is the $\otimes$-multiplicative identity and that the other relevant axioms for a commutative semiring hold, we have that $(\mathcal{T}, \oplus, \otimes, 1_{\mathcal{T}})$ is a commutative semiring

Example: Restriction preorder for tensors

- Let $T \in \mathbb{F}^{d_1 \times d_2 \times d_3}$ and $T' \in \mathbb{F}^{d'_1 \times d'_2 \times d'_3}$ be tensors
- We say that T' **restricts to** T and write $T \leq T'$ if there exist matrices $A \in \mathbb{F}^{d_1 \times d'_1}$, $B \in \mathbb{F}^{d_2 \times d'_2}$, and $C \in \mathbb{F}^{d_3 \times d'_3}$ such that for all $i_1 \in [d_1]$, $i_2 \in [d_2]$, and $i_3 \in [d_3]$ we have

$$T_{i_1, i_2, i_3} = \sum_{i'_1 \in [d'_1], i'_2 \in [d'_2], i'_3 \in [d'_3]} A_{i_1, i'_1} B_{i_2, i'_2} C_{i_3, i'_3} T'_{i'_1, i'_2, i'_3}$$

- The restriction relation $\leq$ is well-defined on $\mathcal{T}$ and can be verified to be a Strassen preorder on $\mathcal{T}$
- Thus, $(\mathcal{T}, \leq)$ is a Strassen semiring
- The rank and asymptotic rank on $(\mathcal{T}, \leq)$ correspond to the standard notions of tensor rank and asymptotic tensor rank

Example: Graphs

- Let $\mathcal{G}$ consist of the isomorphism classes of (nonempty) undirected graphs
- For two graphs G and H, let $G \sqcup H$ be the **disjoint union** and let $G \otimes H$ be the **strong product** of G and H
- (The strong product is essentially obtained from the Kronecker product of the adjacency matrices of G and H.)
- The set $\mathcal{G}$ equipped with $\sqcup$ and $\otimes$ is a commutative semiring with the one-vertex graph K_1 as the multiplicative identity

Example: Cohomomorphism preorder for graphs

- For a graph G, let us write $\bar{G}$ for the complement graph of G
- For two graphs G and H, a mapping $\phi : V(G) \to V(H)$ is a **homomorphism** from G to H if for every $u, v \in V(G)$ we have that $\{u, v\} \in E(G)$ implies $\{\phi(u), \phi(v)\} \in E(H)$
- We say that two graphs G and H **cohomomorphic** and write $G \leq H$ if there exists a homomorphism from $\bar{G}$ to $\bar{H}$
- The cohomomorphism relation $\leq$ is a Strassen preorder on $\mathcal{G}$ and thus $(\mathcal{G}, \leq)$ is a Strassen semiring
- For a graph G, the rank $R(G) = \bar{\chi}(G)$ is the clique cover number and the subrank $Q(G) = \alpha(G)$ is the independence number of G
- The asymptotic subrank $\tilde{Q}(G) = \Theta(G)$ is the **Shannon capacity** of G
- Perfect graphs G satisfy $\alpha(G) = \Theta(G) = \bar{\chi}(G)$;
  the Shannon capacity of C_5 is known but open already for all longer odd cycles

A (very) high-level sketch of Strassen duality

- A Strassen preorder P admits a **closure** $\tilde{P} \supseteq P$ that is also a Strassen preorder; closure satisfies $\tilde{\tilde{P}} = \tilde{P}$; when $\tilde{P} = P$ we say that P is **closed**
- **Theorem.** (Structure of Closed Strassen Preorders)
  Every closed Strassen preorder P is the intersection of all total closed Strassen preorders that extend P
- The (integral) notions of rank and subrank for P admit extensions to **fractional rank** and **fractional subrank** with $Q(a) \leq \kappa(a) \leq \rho(a) \leq R(a)$
- For a total closed Strassen preorder $Q \supseteq P$ it holds that $\kappa_Q = \rho_Q$;
  the map $\phi = \kappa_Q = \rho_Q$ is thus a **monotone homomorphism** of P
- The set of all monotone homomorphisms of P form its **asymptotic spectrum** $\mathcal{X}$
- **Theorem.** (Duality for asymptotic rank)
  For all $a \in \mathcal{R}$ it holds that $\tilde{R}(a) = \sup\{\phi(a) : \phi \in \mathcal{X}\}$

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