$\exists\mathbb{R}$-completeness in geometric algorithms

Sándor Kisfaludi-Bak

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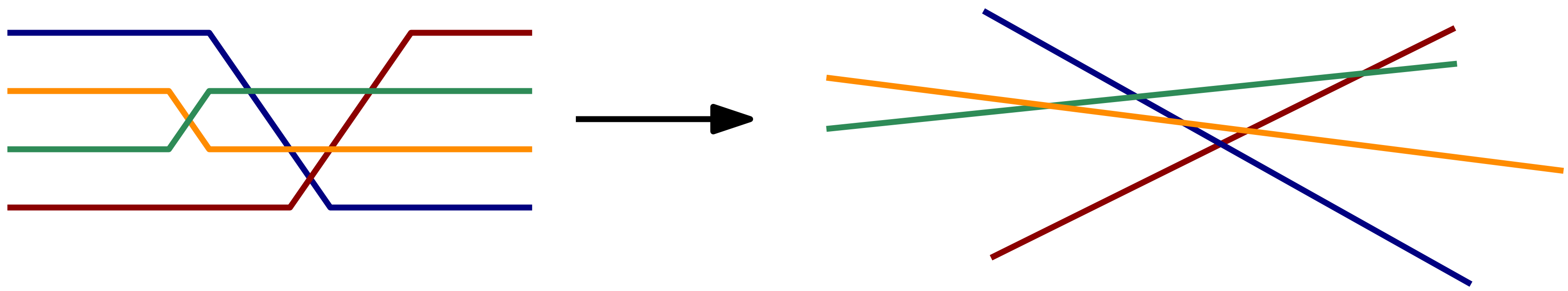
HIIT | HELSINKI INSTITUTE FOR INFORMATION TECHNOLOGY

Aalto University School of Science

NP-hard, but is it in NP?

1. STRETCHABILITY: given a set of n pseudolines (x-monotone curves with exactly 1 crossing between any pair), decide if it is homeomorphic to an arrangement of lines.



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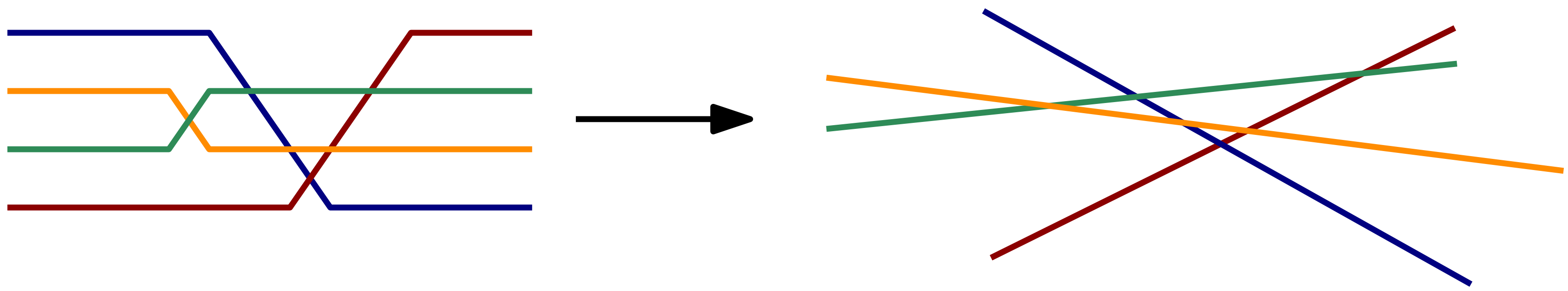

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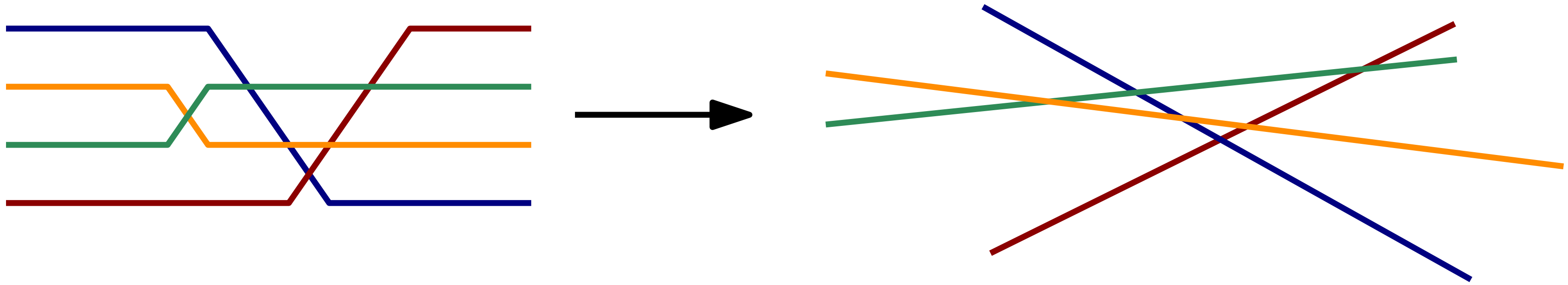

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Double exponential precision is required
$\Rightarrow$ the "natural" witness has exponentially many bits!

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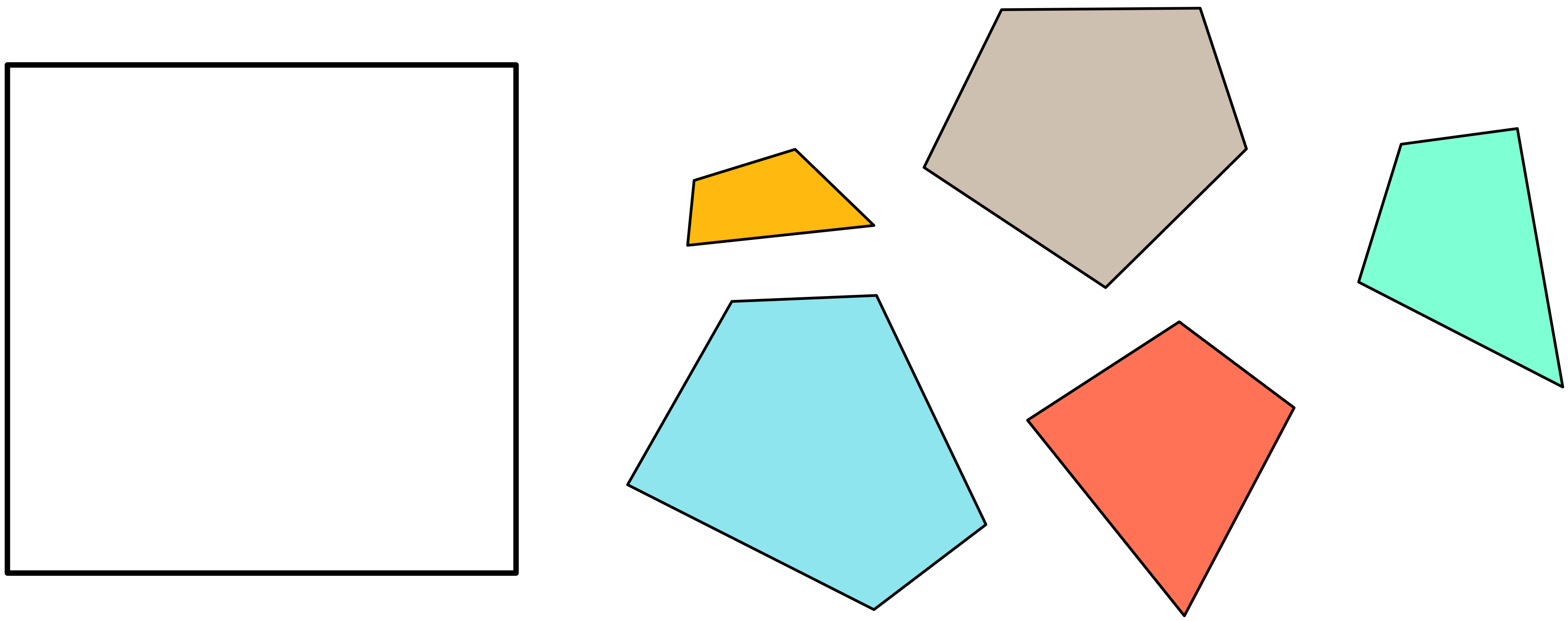

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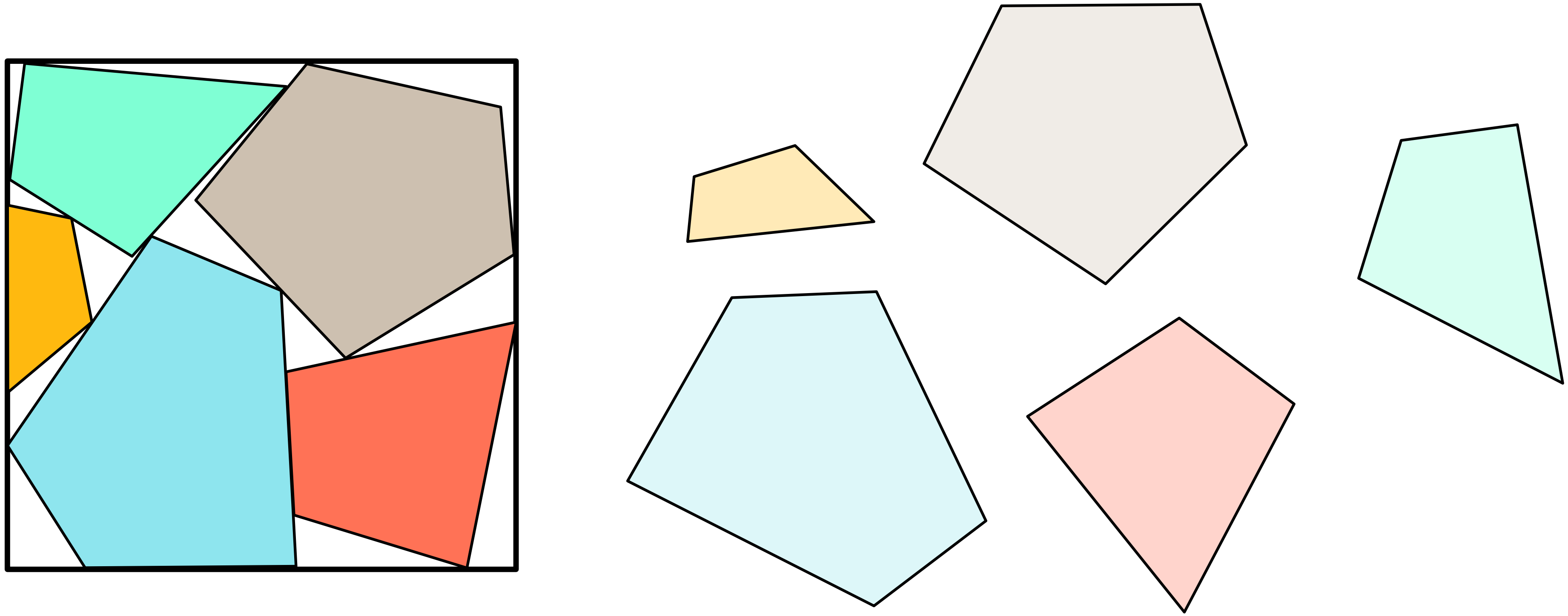

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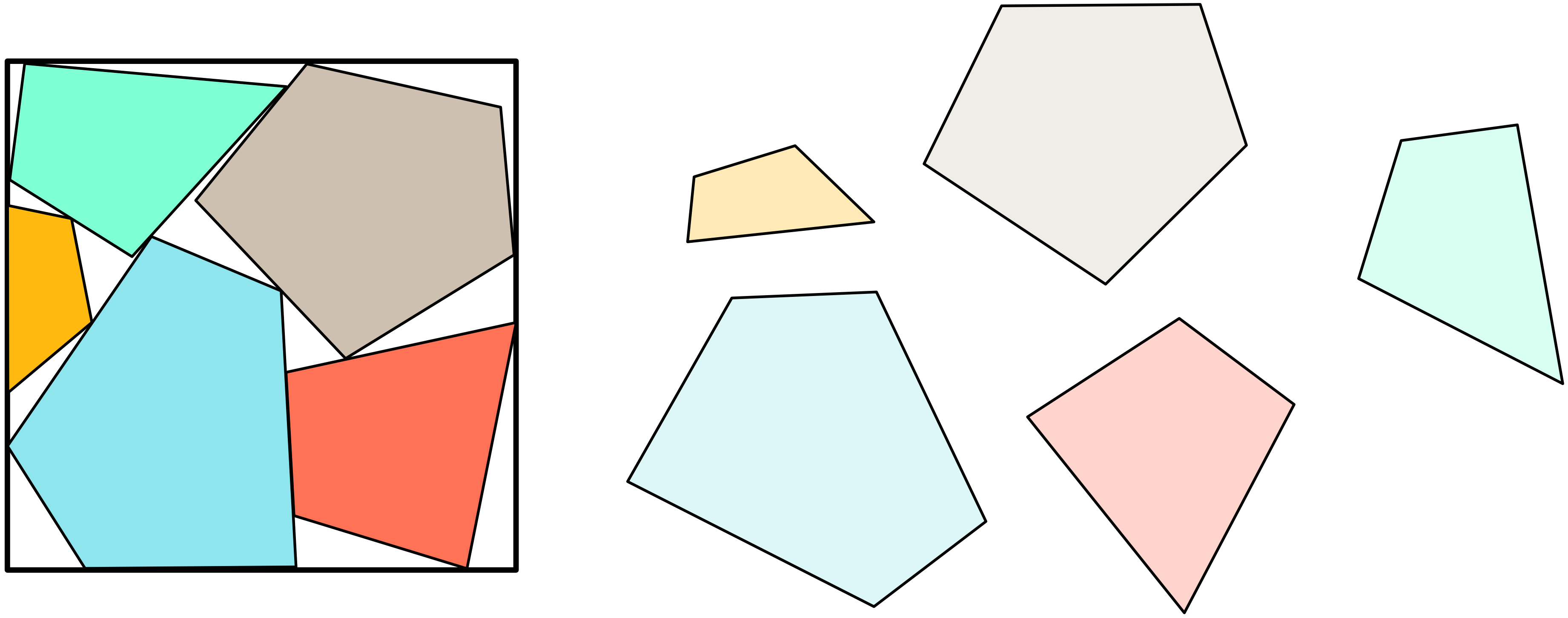

Scary: Are we sure that the coordinates of the rotated translated polygons can be expressed with $O(n)$ bits?

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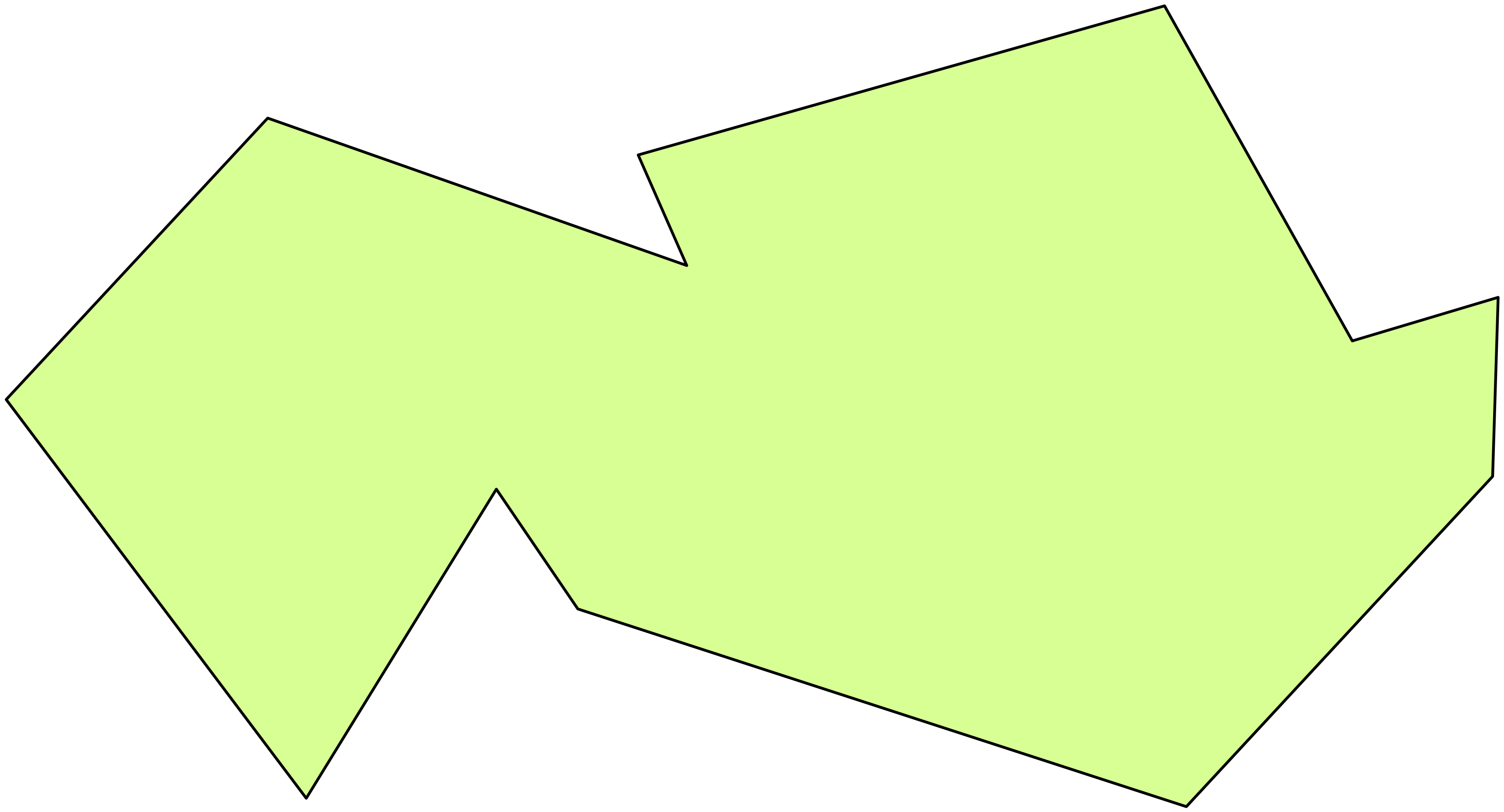

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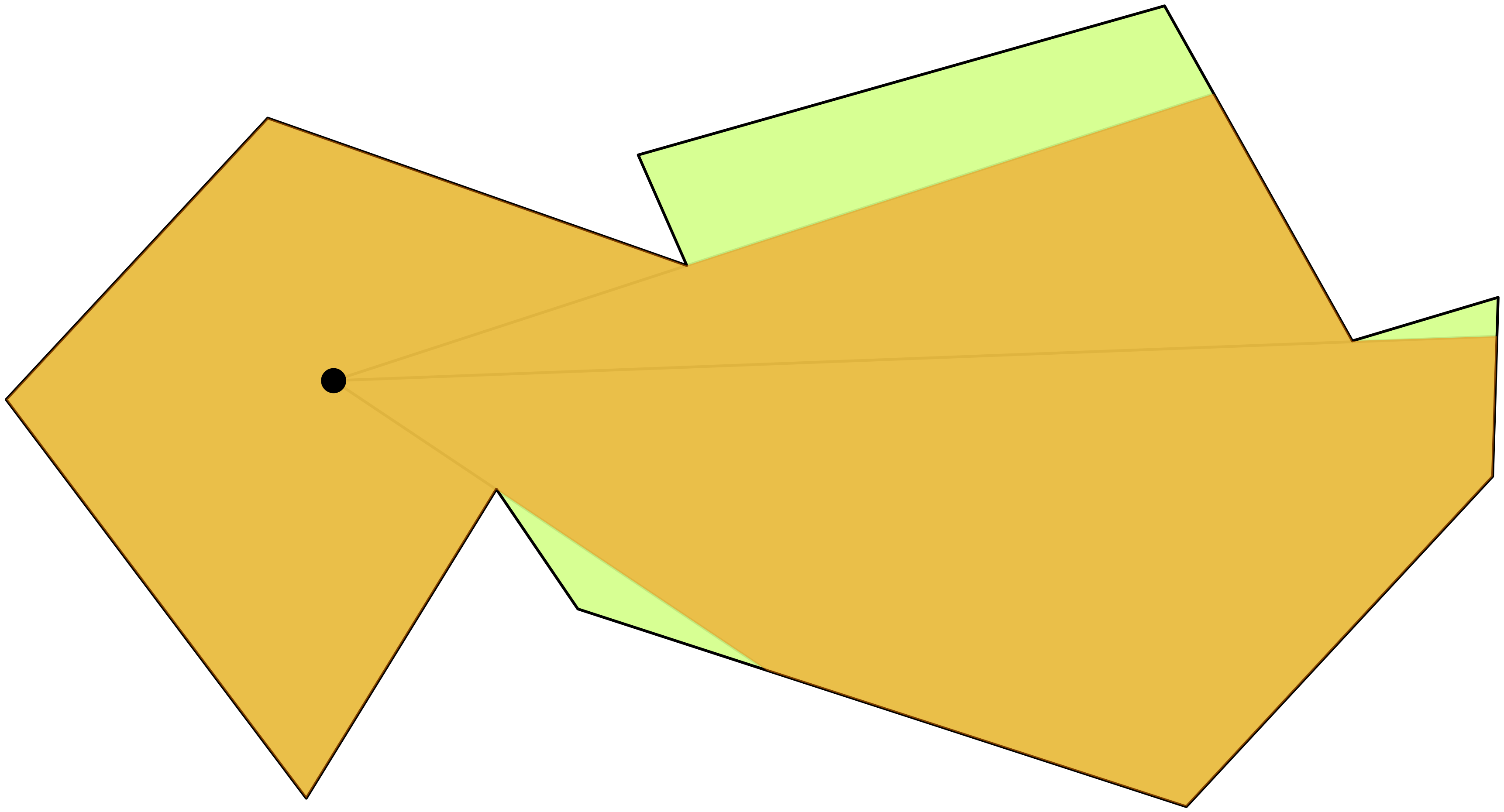

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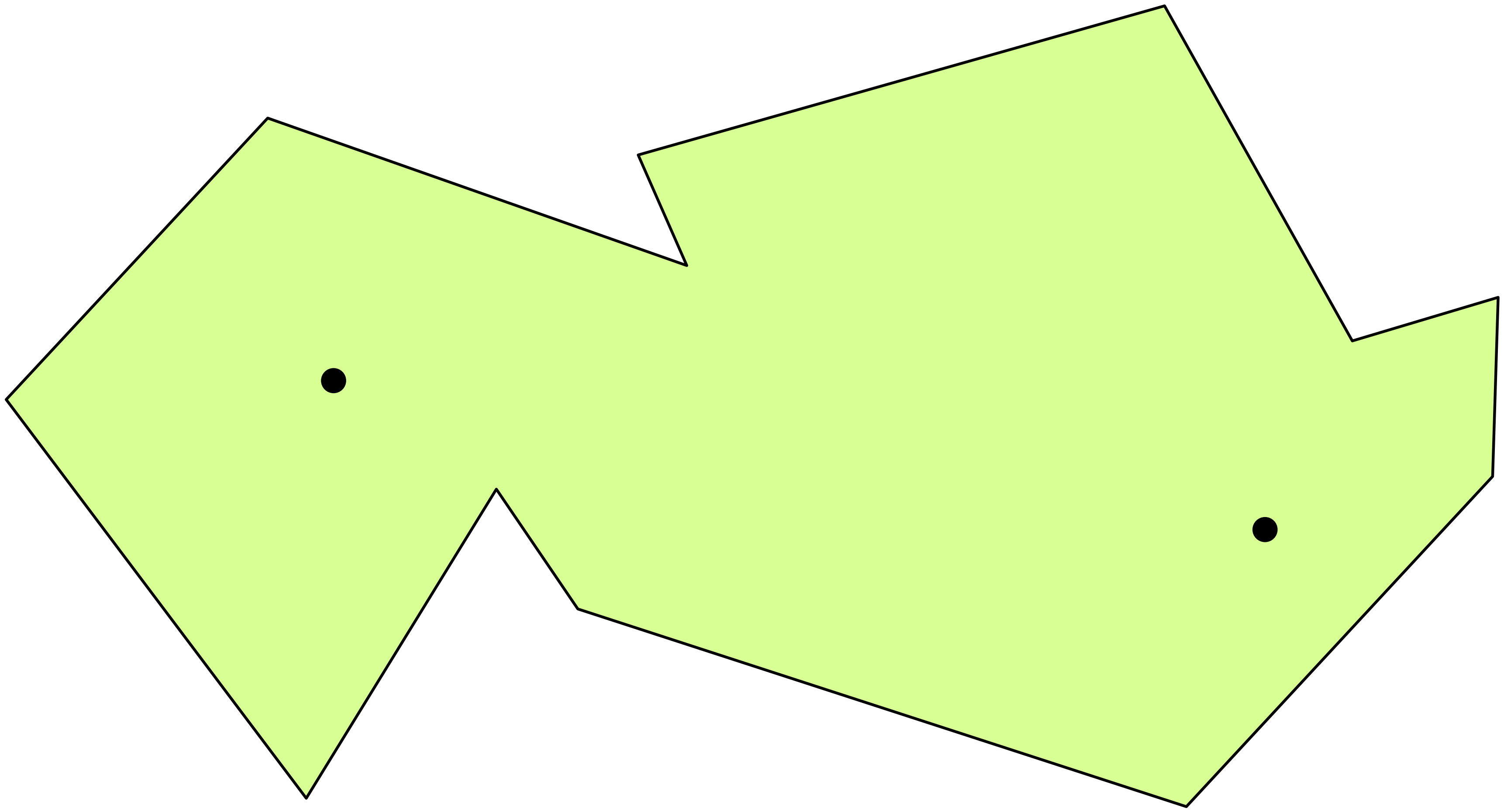

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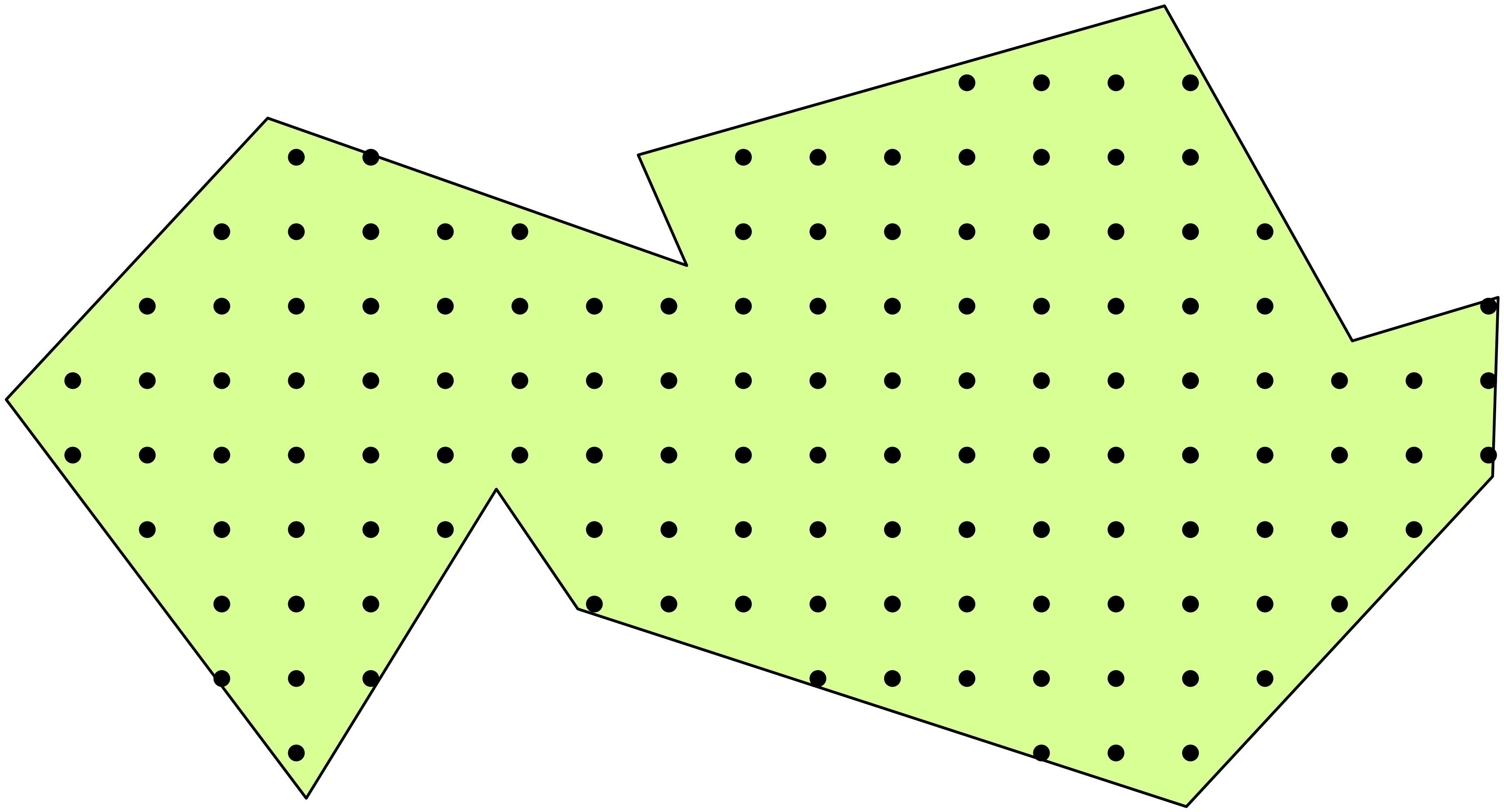

Perhaps we can always find valid guards on a fine grid?
That would guarantee NP-membership...

Further examples: NP-hard but witness issues

UNIT DISK GRAPH RECOGNITION: Given a graph G, is it realizable as an intersection graph of unit disks?

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Unit Disk Graph Recognition: Given a graph G, is it realizable as an intersection graph of unit disks?

Geometric Embeddability of Complexes: Given a d-dimensional simplicial complex, can it be embedded with geometric simplices into $\mathbb{R}^d$? or into $\mathbb{R}^{d+1}$?

Convex cover: Given a polygon, can it be covered by k convex pieces of itself?

Existential Theory of the Reals

First order theory of the reals is the set of true sentences with the symbols

$$\{x_1, x_2, \ldots, \forall, \exists, \wedge, \vee, \neg, 0, 1, +, -, \cdot, (,), =, \leq, <\}$$

where x_i are variables over the real numbers.

Easy to get integer constant k with O(log k) formula length.

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where Φ is a quantifier-free formula.

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$\exists\mathbb{R}$: the class of problems poly-time reducible to ETR

Examples and non-examples

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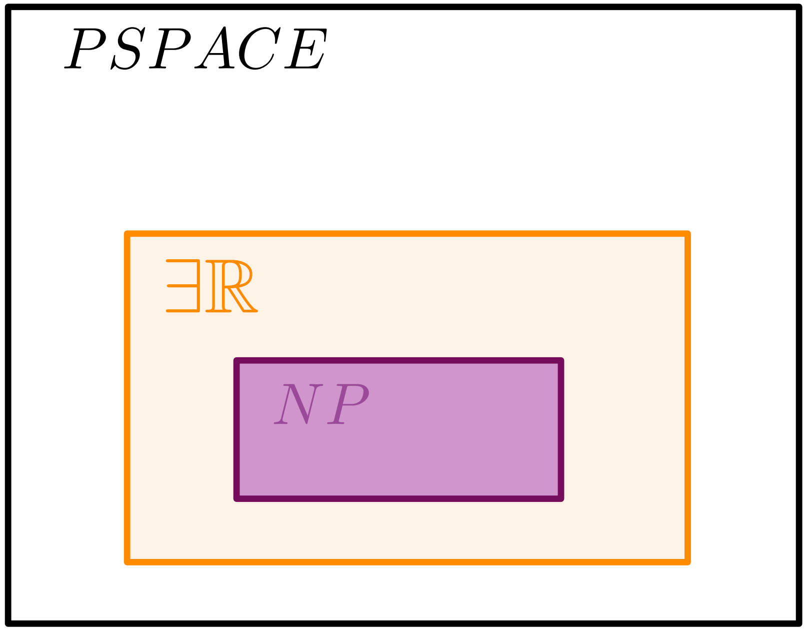
- The squared distances of vertices in the same polygon are fixed.
- Each vertex of polygon i is separated from polygon i' by the line of one of the sides of i' for all $i \neq i'$.

The complexity landscape



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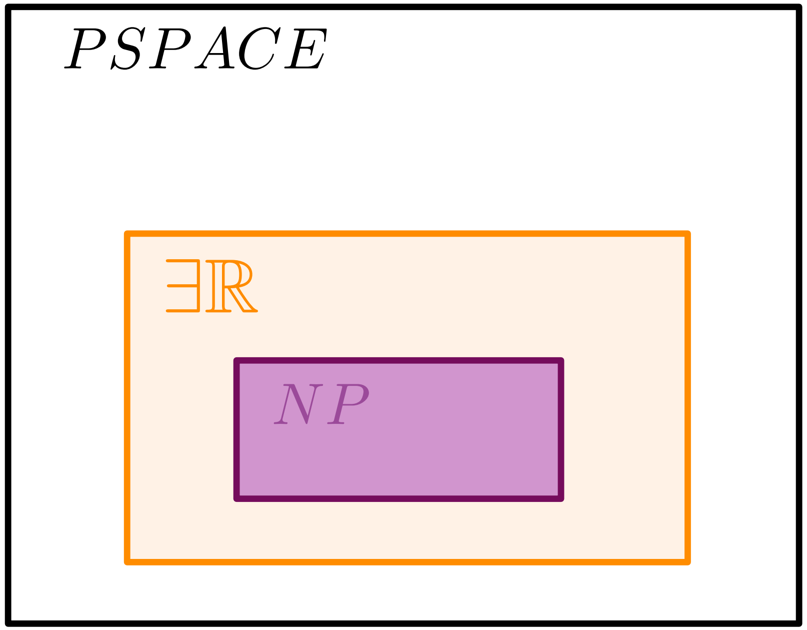

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Easy!

Take a 3-SAT instance, and regard as an ETR formula. To restrict each variable $x \in \mathbb{R}$ to be Boolean, add the conditions:

$$(x \cdot x = x)$$

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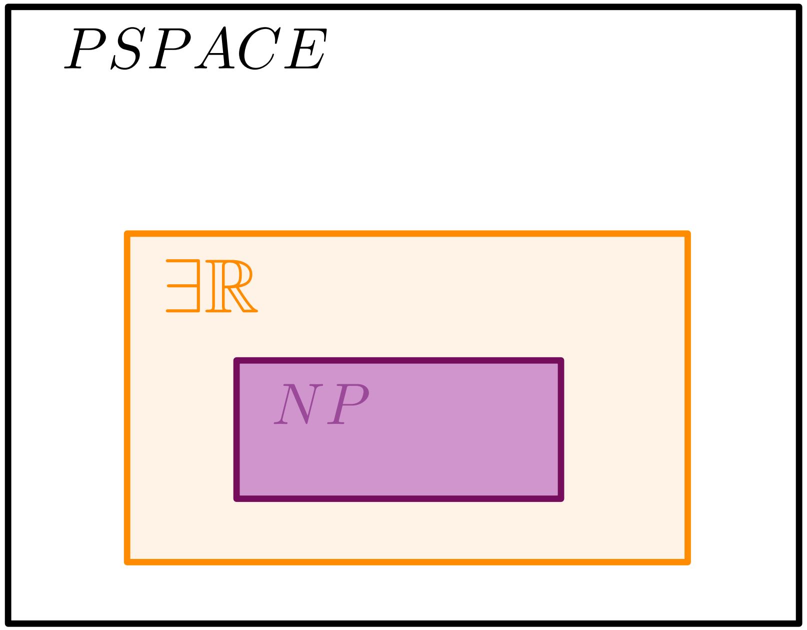

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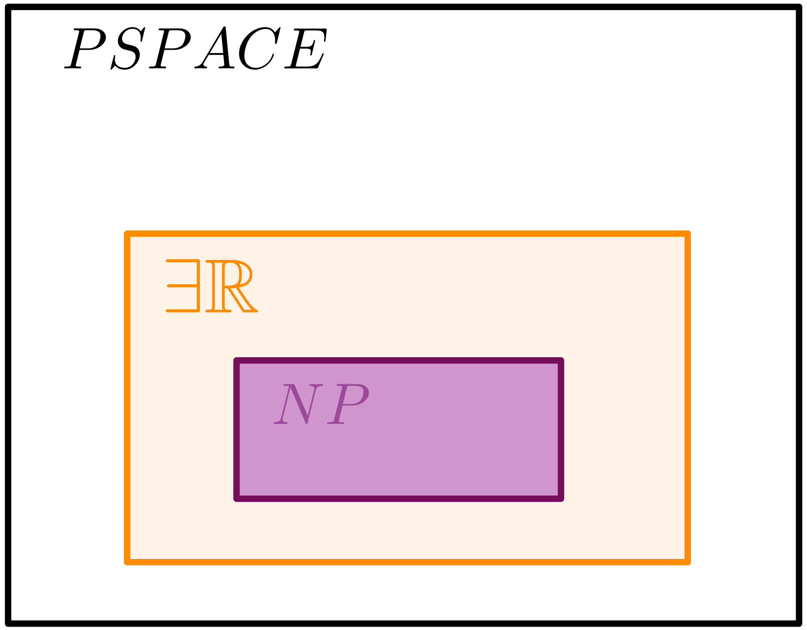

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Many believe that $NP \neq \exists\mathbb{R}$, and take an $\exists\mathbb{R}$-hardness proof as evidence that the problem is likely not in NP.

Smarter $\exists\mathbb{R}$ containment proofs

Theorem (Erickson, van der Hoog, Miltzow, FOCS 2020), ***roughly stated***
A problem is in $\exists\mathbb{R}$ iff it has a *real verification algorithm*, i.e., can be verified in poly time on a real RAM with a polynomial witness (of reals and integers).

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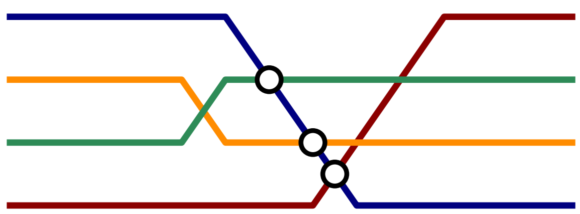
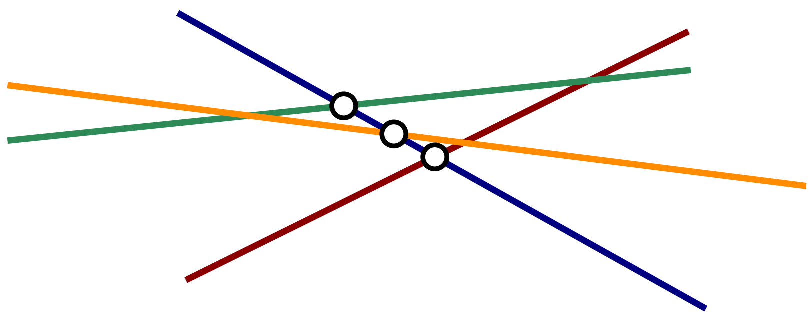
$\textsc{Stretchability} \in \exists\mathbb{R}$ because:

Witness: pairwise intersection points of each line pair (i,j) given as pairs of real coordinates $(x_{ij}, y_{ij}) \in \mathbb{R}^2$.

Algorithm to verify: check using real arithmetic that the points on each line i are indeed collinear, and that they have the same ordering as the intersections of the corresponding curve in the pseudoline arrangement.





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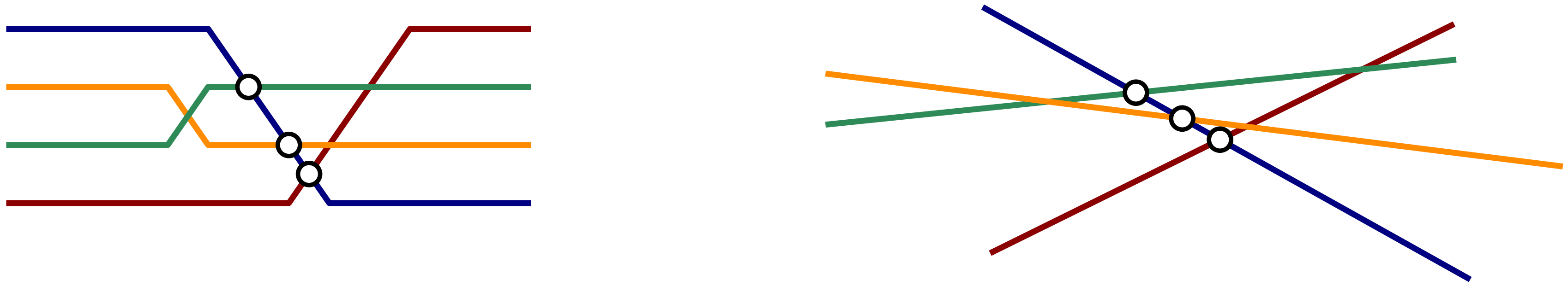

ART GALLERY $\in \exists\mathbb{R}$ because:

Witness: Guard coordiantes as real numbers

Algorithm to verify: Compute using real arithmetic the polynomial complexity region seen by each guard, and check if their union covers the gallery.

A glimpse of the $\exists\mathbb{R}$-hardness for Stretchability

Theorem (Mnëv '88, Shor '91), *very roughly stated*
For each ETR formula there is an equivalent ETR formula (defining a "topologically equivalent" semi-algebraic set) whose variables are > 1 and have a fixed strict ordering, and where each equation is a simple addition or a simple multiplication.

Basically, we can assume that:

$1 < x_1 < x_2 < \cdots < x_n,$ and formulas are either $x_i + x_j = x_k$ or $x_i \cdot x_j = x_k$

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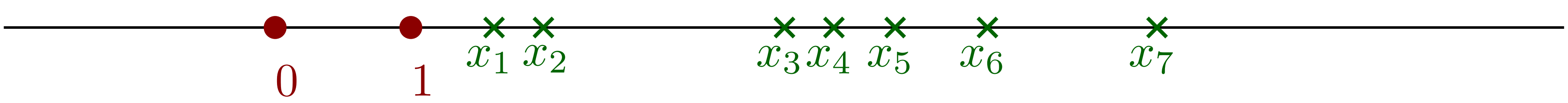
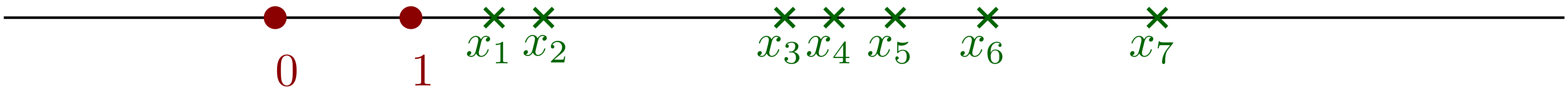
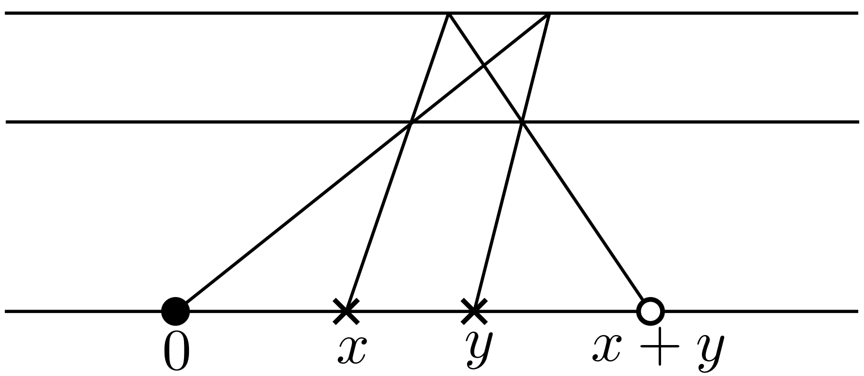
Idea: put the variables and $0, 1$ on a line, so lengths are represented by distance to the 0 point.
Use projective geometry gadgets for addition and multiplicaiton.



Addition and multiplication with cross-ratio



$x + y$



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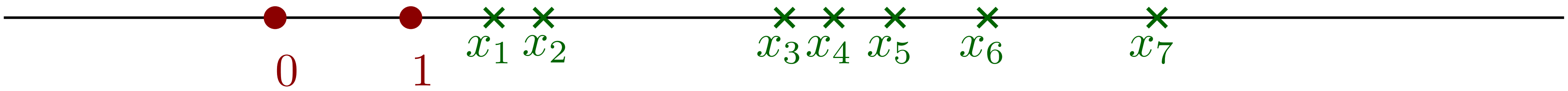

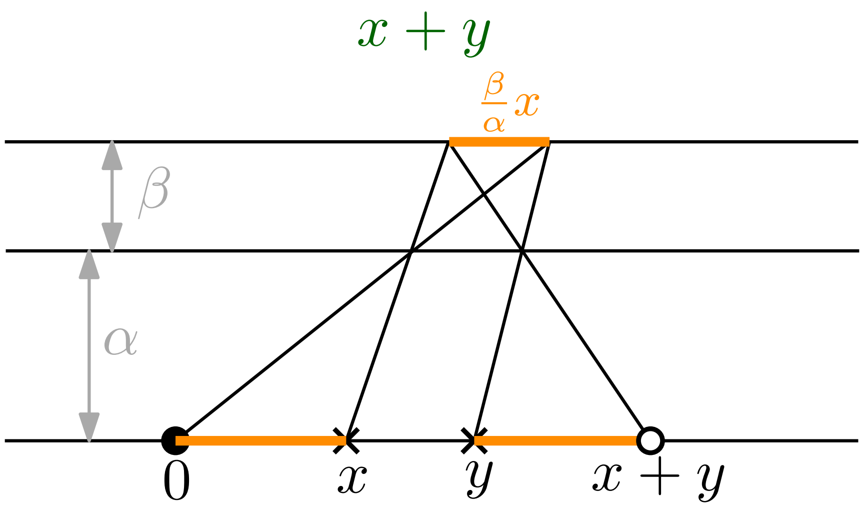

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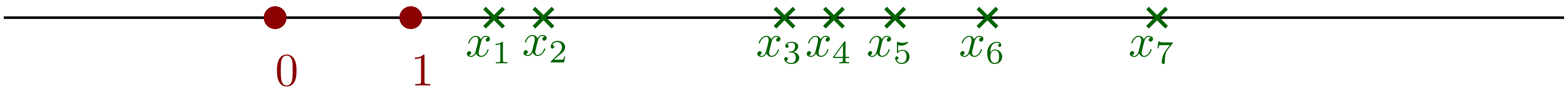

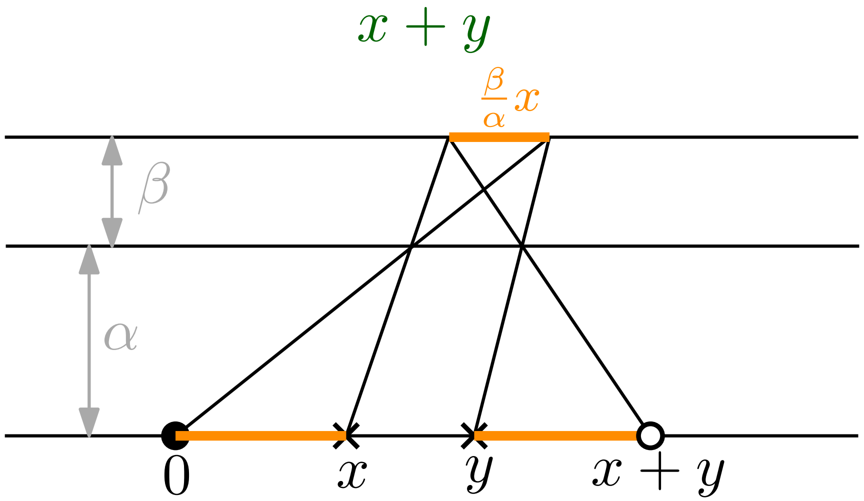

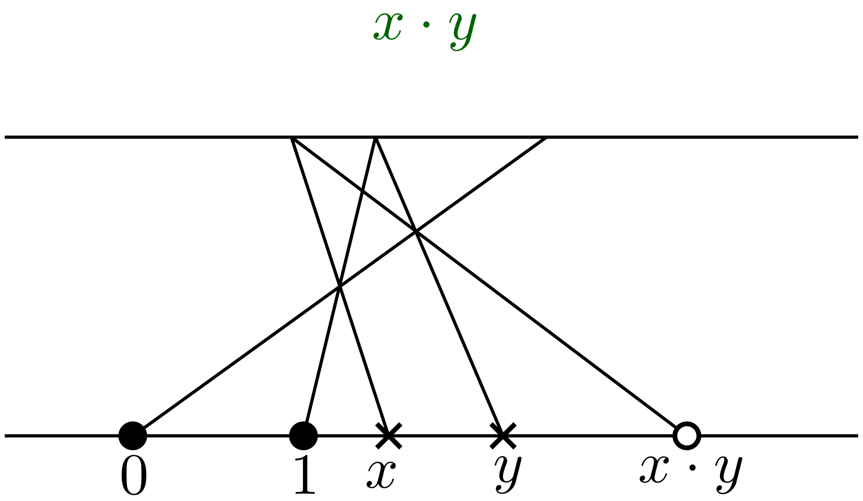

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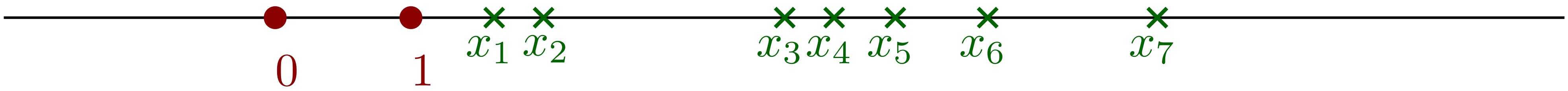

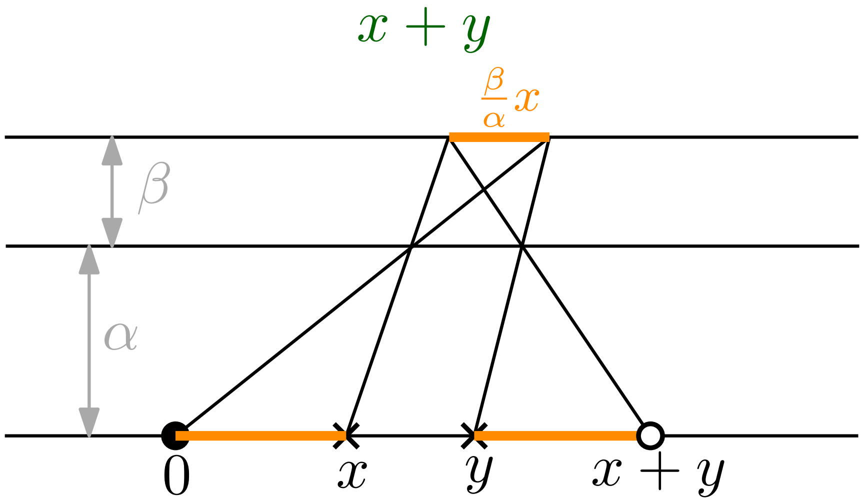

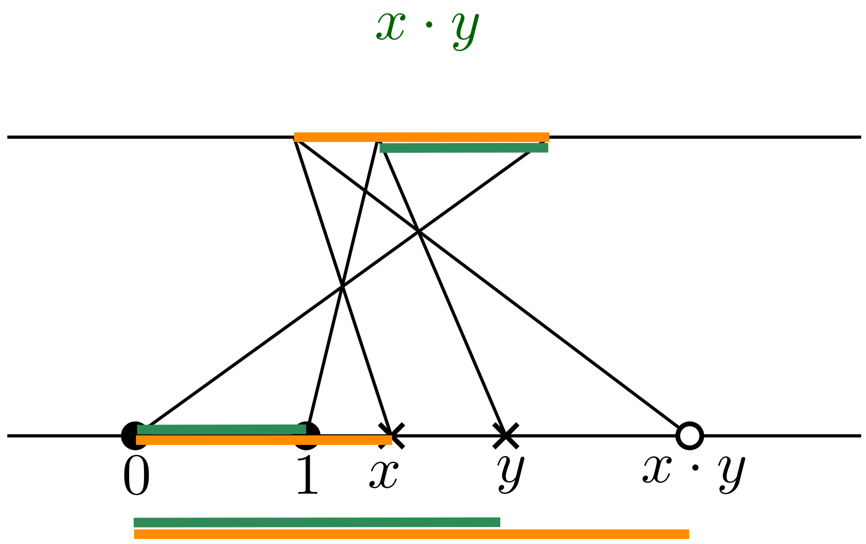

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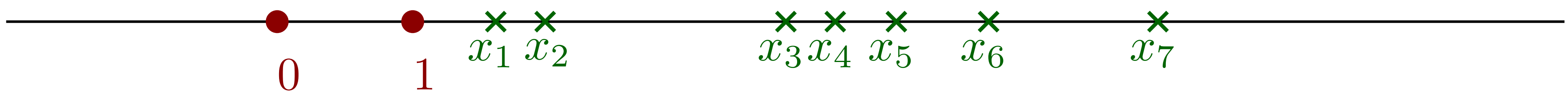
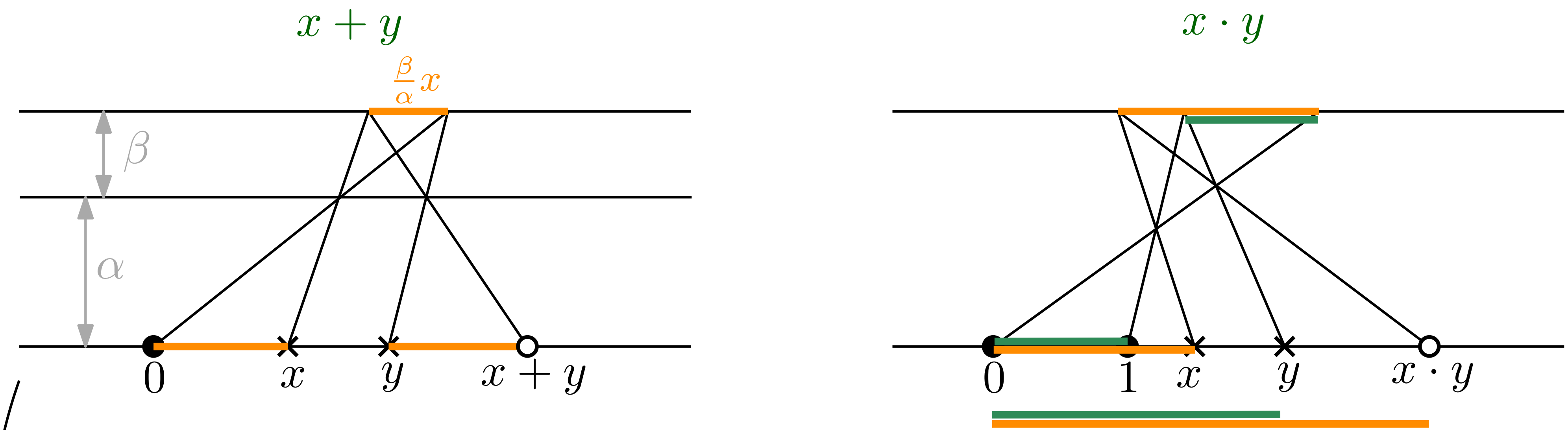
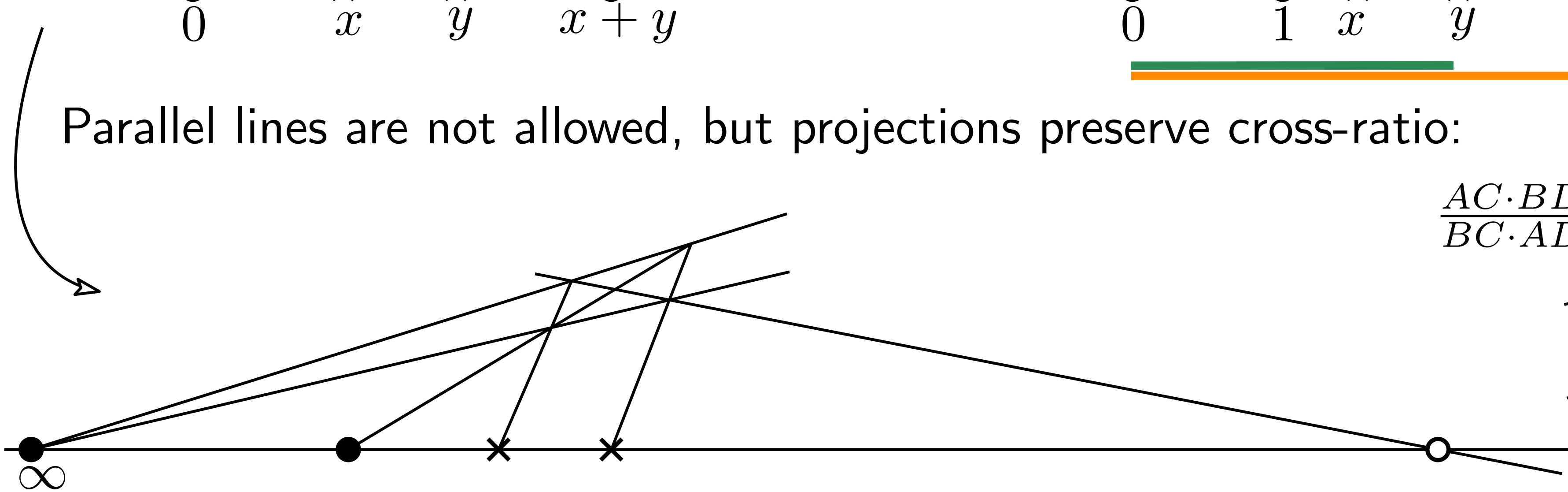
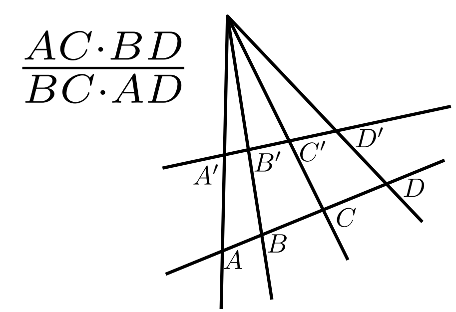




Parallel lines are not allowed, but projections preserve cross-ratio:





When should you suspect $\exists\mathbb{R}$-completeness?

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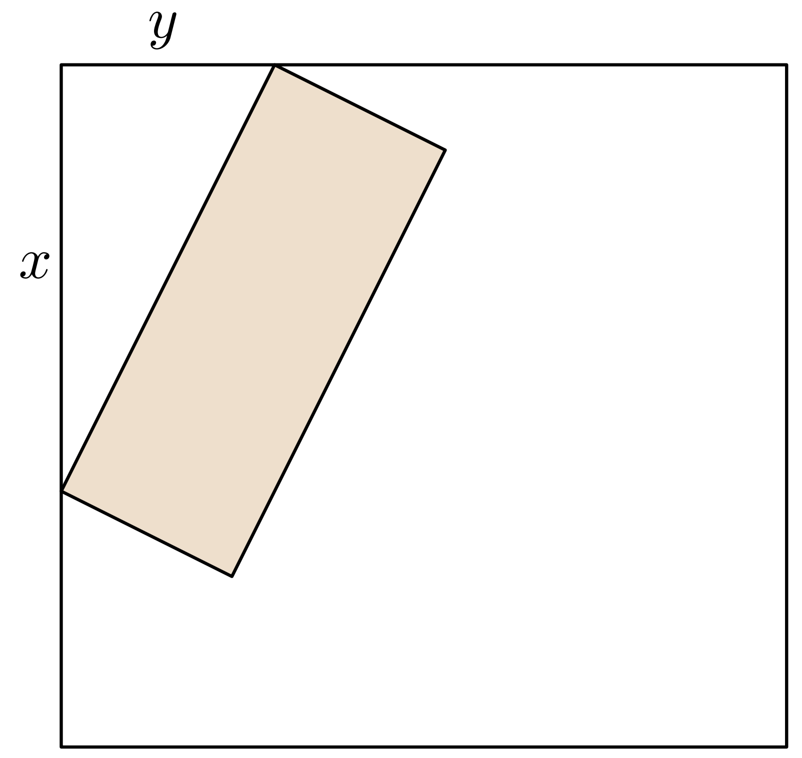

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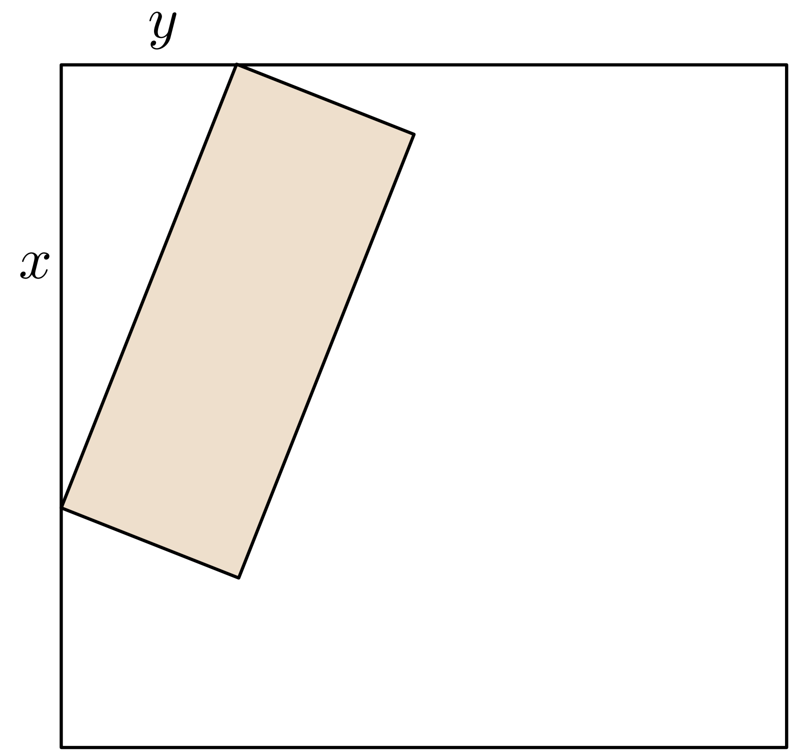

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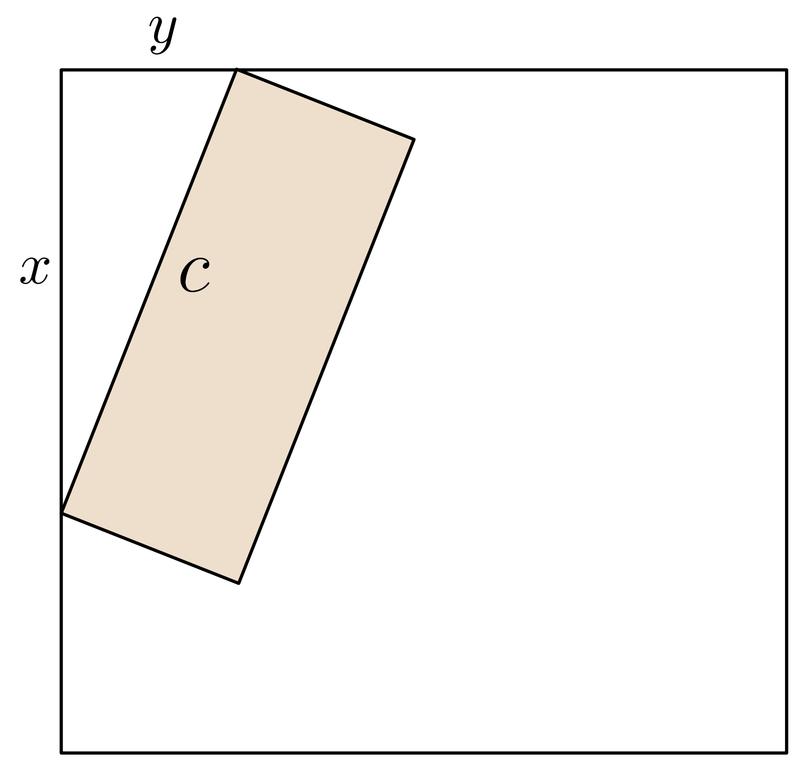

- Non-linear behaviour

$$x = \sqrt{c^2 - y^2}$$

∃ℝ-compelte ETR variants

ETR-Square:

$x = 1 \quad x + y = z \quad x^2 = y$ and $x_i \in [-1, 1]$ for all i.

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ETR-INV:

$x = 1 \quad x + y = z \quad x \cdot y = 1$ and $x_i \in [\frac{1}{2}, 2]$ for all i.

$\exists\mathbb{R}$-compelte ETR variants

ETR-Square:
$x = 1 \quad x + y = z \quad x^2 = y$ and $x_i \in [-1, 1]$ for all i.

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ETR-f: for some fixed non-linear f:
$x = 1 \quad x + y = z \quad x \geq 0 \quad f(x, y) = 0$ and $x_i \in [-1, 1]$ for all i

Theorem (Miltzow and Schmiermann FOCS'21), roughly
ETR-f is $\exists\mathbb{R}$-complete whenever f is "curved".

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Definition 6 (Well-behaved)**.** We say a function $f : U^2 \to \mathbb{R}$ is *well-behaved* around the origin if the following conditions are met.

- f is a C^2-function, with $U \subseteq \mathbb{R}$ being a neighborhood of $(0, 0)$,
- $f(0, 0) = 0$, and all partial derivatives f_x, f_y, f_{xx}, f_{xy} and f_{yy} are rational, in $(0, 0)$.
- $f_x(0, 0) \neq 0$ or $f_y(0, 0) \neq 0$,
- $f(x, y)$ can be computed on a real RAM [51].

Note that if $p(x, y)$ is a polynomial of the form $\sum_{i,j} a_{i,j} x^i y^j$, then p is well-behaved if and only if $a_{0,0} = 0$, $a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}$ are rational, and ($a_{1,0} \neq 0$ or $a_{0,1} \neq 0$).

Definition 7 (Curved)**.** Let $f : U^2 \to \mathbb{R}$ be a function that is well-behaved around the origin. We write the curvature of f at zero by

$$\kappa = \kappa(f) = \left(\frac{f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}} \right) (0, 0),$$

see Figure 4 for an illustration. We say f is

- **curved** if $\kappa(f) \neq 0$,

Beyond $\exists\mathbb{R}$: $\forall\mathbb{R}$, $\forall\exists\mathbb{R}$, ...

$\exists\mathbb{R}$: the class of problems poly-time reducible to some formula given formula
$\exists x_1 \ldots \exists x_k \quad \Phi(x_1, \ldots, x_k)$

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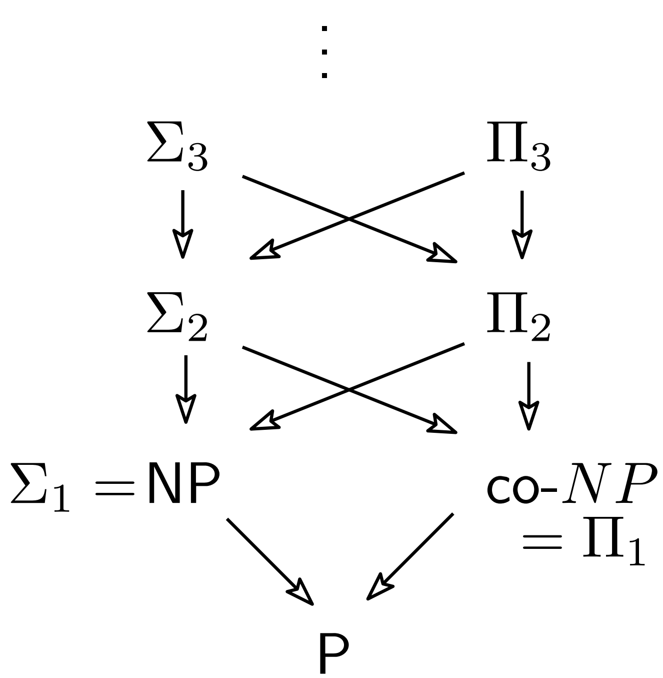

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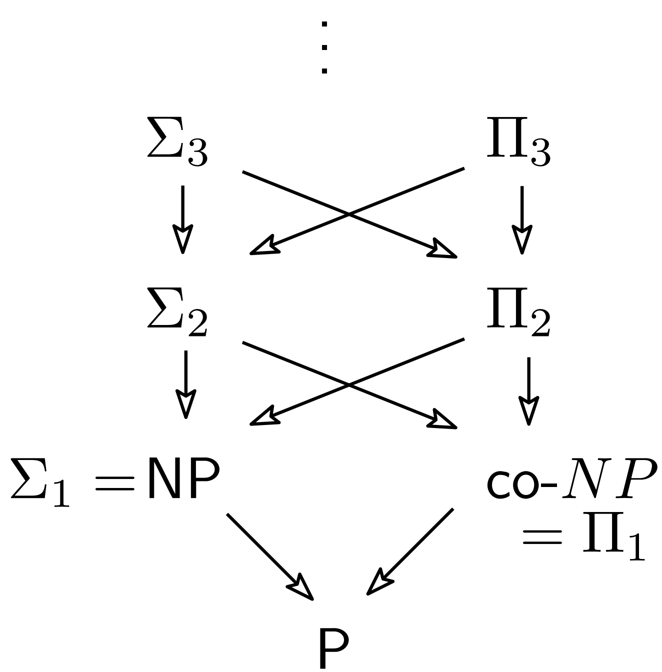
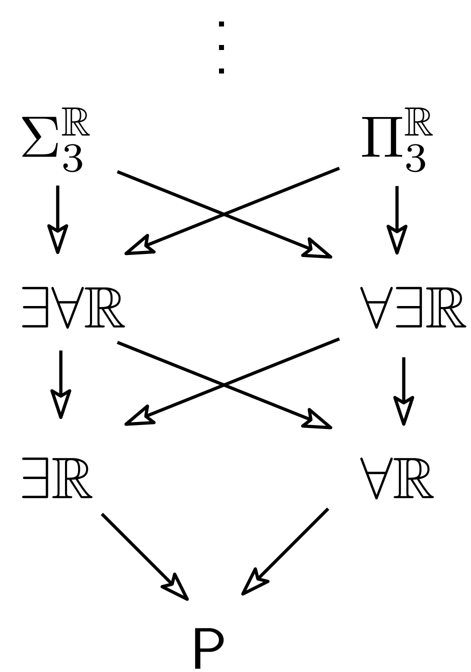
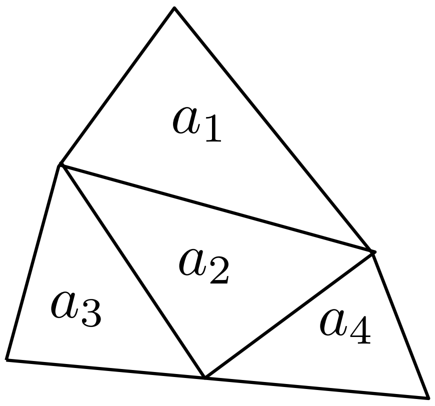




Complete problems for the 2nd level of the hierarchy?

AREA UNIVERSALITY: Given a plane graph, can all positive area-assignments to its bounded faces be realized with a straight-line drawing?



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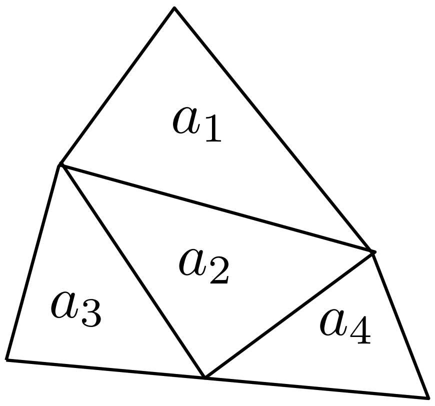

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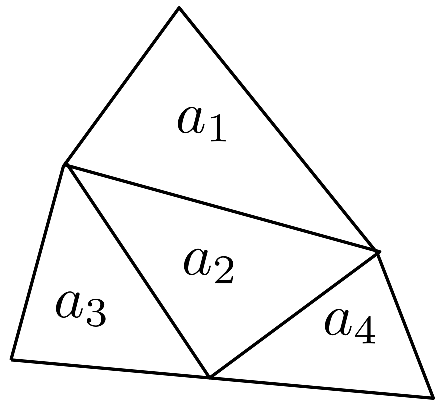

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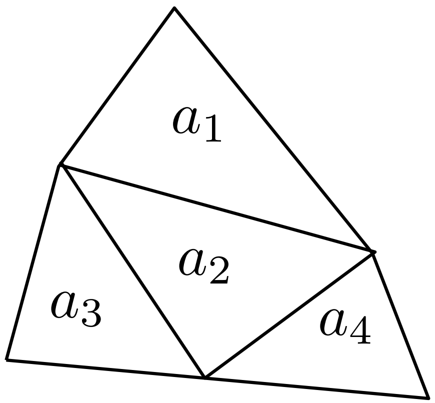

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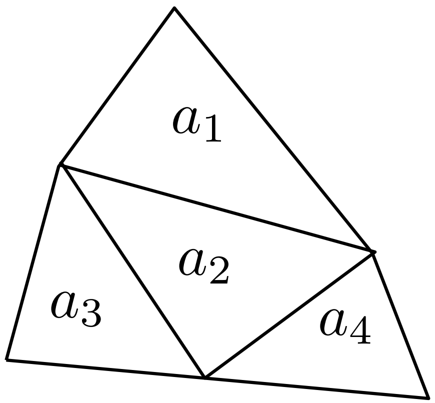

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Theorem (Jungeblut, Kleist, Miltzow '23)
HAUSDORFF DISTANCE is $\forall\exists_<\mathbb{R}$-complete.
Even when both sets are defined by a single polynomial equation of degree ≤ 4.

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How to solve $\exists\mathbb{R}$-complete problems anyway?

Theorem (Basu, Pollack, Roy JACM'96),
Algorithm for the first order theory of the reals
Given a set of s polynomial inequalities (<0,>0,=0) of degree at most d in k variables, where the variables are alternatingly quantified in ω blocks, having $k_1, k_2, \ldots, k_\omega$ variables ($\sum k_i = k$), the sentence can be decided with $s^{\prod(k_i+1)} d^{\prod O(k_i)}$ arithmetic operations.

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... But Canny's algorithm is not much slower and uses only poly space.

Not so geometric $\exists\mathbb{R}$-complete examples

TRAINING NEURAL NETWORKS (*Abrahamsen, Kleist, Miltzow '21*):
Given a neural network architecture (DAG with s input and t output neurons) and data set ($D \in \mathbb{R}^{s+t}$ giving ground truth), an activation function for each neuron, and a cost funciton $c : \mathbb{R}^t \times \mathbb{R}^t \to \mathbb{R}_{\geq 0}$ that is 0 on the diagonal, are there weights such that the total cost of the error $\sum_{d \in D} c(y(d), y'(d)) < \delta$?

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LOW-RANK MATRIX COMPLETION (*Bertsimas, Cory-Wright, Pauphilet '21*)

$$\min_{X \in \mathbb{R}^{n \times m}} \langle C, X \rangle + \lambda \mathrm{rk}(X) \text{ s.t. } AX = B, \mathrm{rk}(X) \leq k, \ X \text{ is positive semidefinite}$$

Euclidean TSP and other uncategorized

EUCLIDEAN TSP: given n points in the plane with integer coordinates, is there a closed curve of length at most x containing all the points?

$\rightarrow$ between NP and $\exists\mathbb{R}$. Lack of continuity!

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Shortest path among 3D obstacles: given pw disjoint open polyhedra with n total vertices, and points $a, b \in \mathbb{R}^3$, is there a path from a to b of length at most x that is disjoint from the polyhedra?

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MINIMUM SPANNING TREE: given n points in the plane with integer coordinates, compute a minimum spanning tree.

→ Can be solved in $O(n \log n)$.

Sum of square roots

Sum of Square Roots

Let $a_1, \ldots, a_k, b_1, \ldots b_k \in \{0, 1, \ldots, n\}$. Decide if

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We want: $\log(1/r(n,k)) = \mathrm{poly}(\log n, k)$.

(Would imply Sum of Square Roots $\in$ P.)

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Deciding if $\sum_{i=1}^{k} \sqrt{a_i} - \sum_{i=1}^{k} \sqrt{b_i} = 0$ is in RP. (Blömer, FOCS'91)
Sum of Square Roots $\in P^{PP^{PP^{PP}}}$ (Allender et al, CCC'06).

Conclusion

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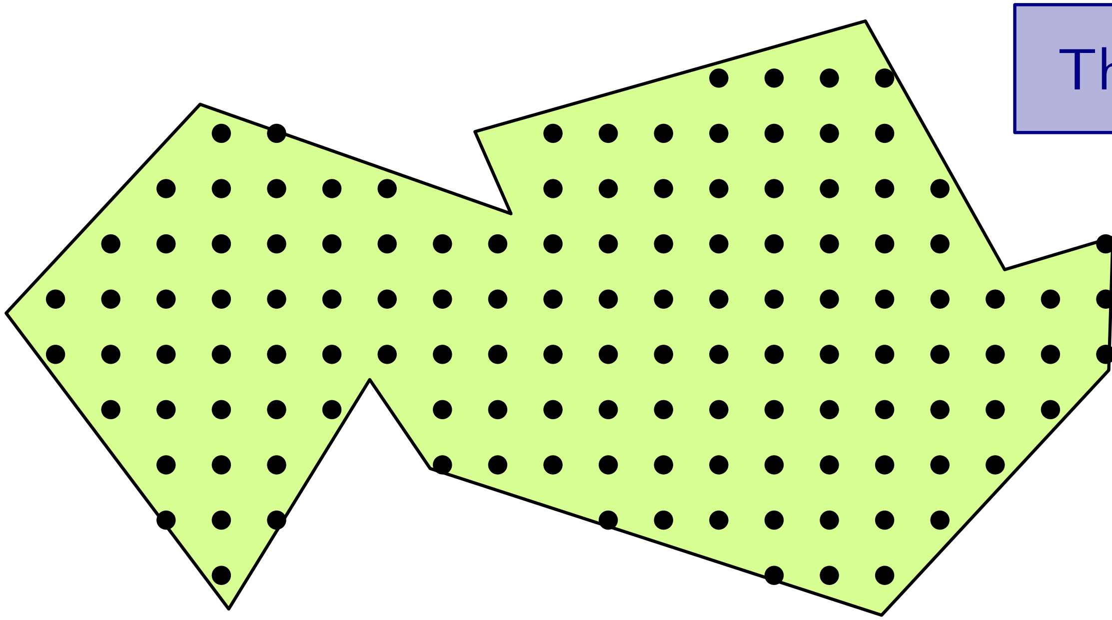

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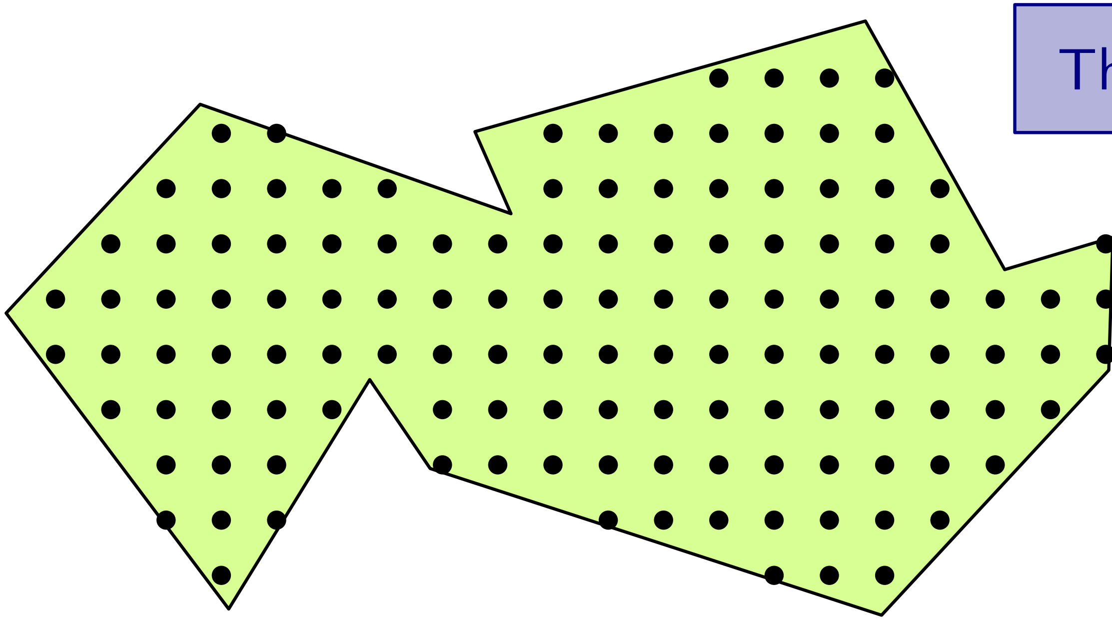

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