The Origins of Fine-Grained Complexity

Massimo Equi

Foundation Friday

26 April 2024

What is Fine-Grained Complexity?

Lower Bounds

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Let's say you can solve problem A in time $\boldsymbol{O(n^2)}$
How to prove that A cannot be solved in $\boldsymbol{O(n^{2-\epsilon})}$?
What can we condition on?

ETH and SETH

Fine-grained complexity provides the hypotheses for that [Impagliazzo and Paturi, 2001]

Exponential Time Hypothesis (ETH)

There exists a constant $\bar{\alpha}$ s.t. no algorithm solves CNF-SAT with n variables in time $O(2^{\alpha n})$, $\alpha < \bar{\alpha}$

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Idea: if we solve problem A in $\boldsymbol{O(n^{2-\epsilon})}$, then we solve CNF-SAT in $\boldsymbol{O(2^{(1-\epsilon')n})}$, contradicting SETH

Some notation

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n variables $v_1, \ldots, v_n$, k clauses $c_1, \ldots, c_q$

$$F(v_1, \ldots, v_n) = (v_1 \vee \bar{v_2}) \wedge (\bar{v_1} \vee v_2 \vee \bar{v_3}) \wedge (v_1 \vee v_3 \vee \bar{v_4}) \wedge \cdots$$

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For all $k \geq 3$, $s_k > 0$

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$$s_\infty = \lim_{k \to \infty} s_k = 1$$

Arguments in favour of ETH

Theorem

The following statements are equivalent:

For every $k \geq 3$, $s_k > 0$ (ETH)

For some k, $s_k > 0$

$s_3 > 0$

SNP $\not\subseteq$ ***SUBEXP***

Satisfiability of linear-sized circuits cannot be solved in subexponential time

[Impagliazzo, Paturi and Zane, 1998]

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[Impagliazzo, Paturi and Zane, 1998]

SNP: the class of properties expressible by a series of second order existential quantifiers, followed by a series of first order universal quantifiers, followed by a basic formula

Note:

- $P \neq NP$ allows e.g. $O(2^{\sqrt{n}})$ for k-SAT, ETH **does not**
- ETH $\Rightarrow P \neq NP$

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SETH states $s_\infty = 1$, but it might be $s_\infty < 1$

Let's look at (b) more closely

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- Then $\boldsymbol{(1 - d/k)s_\infty < (1 - d/k')s_\infty}$

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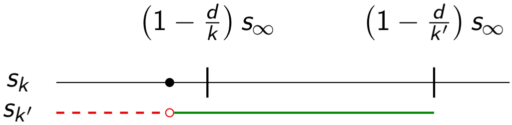

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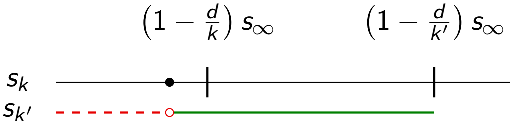

Then $\boldsymbol{s_k < s_{k'}}$, i.e. sequence $\boldsymbol{\{s_k\}}$ is strictly increasing

SETH

Summing up, we know that

- $s_\infty \leq 1$
- $s_k \leq (1 - d/k)s_\infty$, for every k
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Hence the claim

$$\text{SETH}: \quad s_\infty = 1$$

Conditional Lower Bounds

Orthogonal Vectors

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Doing so allows us to use OV instead of CNF-SAT for proving polynomial conditional lower bounds, making the reductions easier

Quick Complexity Background

Orthogonal Vectors (OV) Problem

Let $X, Y \subseteq \{0,1\}^d$ be two sets of n binary vectors of length d.

Determine whether there exist $x \in X, y \in Y$ such that $x \cdot y = 0$

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No algorithm can solve *Orthogonal Vectors* in time $O(n^{\alpha}\,\mathrm{poly}(d))$, $\alpha < 2$

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CNF-SAT can be reduced to OV, thus SETH $\Rightarrow$ OVH

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$$2^{\frac{n}{2}} \left\{ \begin{array}{lll} x_1 = & (0\ \ 0) & (0\ \ 0\ \ 1\ \ 1\ \ 1) & = u_1 \\ x_2 = & (0\ \ 1) & (0\ \ 1\ \ 1\ \ 0\ \ 0) & = u_2 \\ x_3 = & (1\ \ 0) & (1\ \ 0\ \ 0\ \ 0\ \ 1) & = u_3 \\ x_4 = & (1\ \ 1) & (1\ \ 0\ \ 1\ \ 0\ \ 0) & = u_4 \end{array} \right\} m \qquad \begin{array}{c} u_i[h] = 0 \\ \Leftrightarrow \\ x_i \models c_h \end{array}$$

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Solving OV in time $O(m^{2-\epsilon}\text{poly}(d))$ implies the following for CNF-SAT:

$$O(2^{\frac{n}{2}(2-\epsilon)}\text{poly}(q)) = O(2^{n(1-\frac{\epsilon}{2})}\text{poly}(q))$$

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We conclude that OVH is true unless SETH is false (SETH $\Rightarrow$ OVH)

Problems with lower bounds

Problems with a reduction from **OV** (or k-OV) that implies a lower bound, unless SETH is false [Williams, 2018]:

- Graph Diameter for unweighted, undirected graphs with n nodes and $O(n)$ edges
- Edit Distance, Longest Common Subsequence, Dynamic Time Warping Distance, Fréchet Distance
- Subtree Isomorphism
- All Pairs Max Flow
- Subset Sum on n integers and target T

Final remarks

No known connections with **APSP** or **3-SUM**

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Reductions from **Formula-SAT** instead of CNF-SAT
[Abboud, Hansen, Williams and Williams, 2016]

- For a problem with upper bound $O(n^d / \log^c n)$, you can show that $c < \bar{c}$, for some constant $\bar{c}$.

Thank you!

References I

[Impagliazzo, Paturi and Zane, 1998]
Russell Impagliazzo, Ramamohan Paturi, Fancis Zane
Which Problems Have Strongly Exponential Complexity?
FOCS 1998

[Impagliazzo and Paturi, 2001]
Russell Impagliazzo, Ramamohan Paturi
On the Complexity of k-SAT
Journal of Computer and System Sciences, 2001

References II

[Abboud, Hansen, Williams and Williams, 2016]
Amir Abboud, Thomas Dueholm Hansen, Virginia Vassilevska Williams, Ryan Williams
Simulating branching programs with edit distance and friends: or: a polylog shaved is a lower bound made
STOC 2016

[Williams, 2018]
Virginia Vassilevska Williams
On some fine-grained questions in algorithms and complexity
ICM 2018