Proving Impossibility Results in Distributed Algorithms and Beyond

Sebastian Brandt
CISPA Helmholtz Center for Information Security

Lower Bounds

How can we prove complexity lower bounds in the LOCAL model?

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Round Elimination

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- sinkless orientation

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Marks' Technique

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The LOCAL Model

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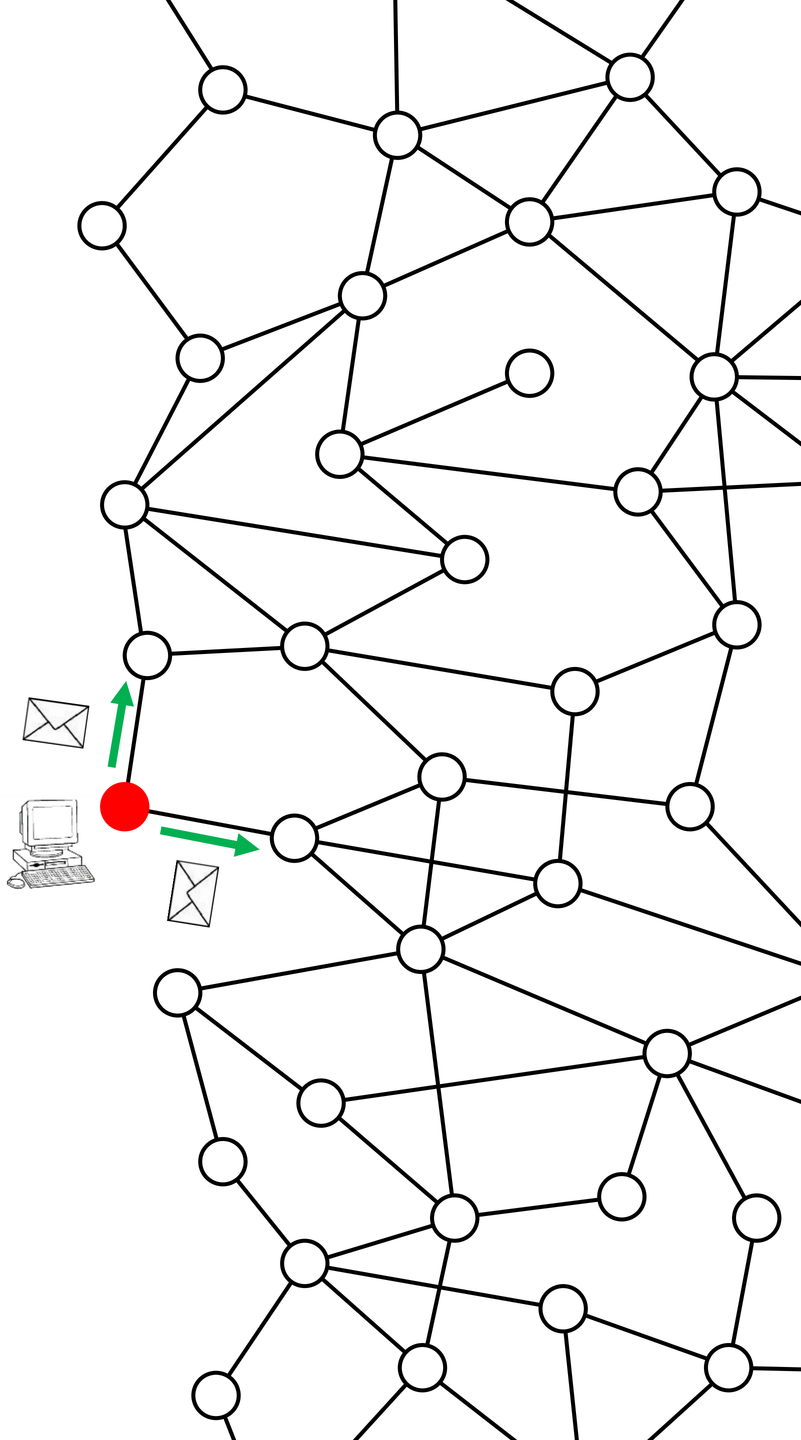

[Linial, FOCS'87]

The LOCAL Model

Synchronous rounds of

1) Communication
2) Computation

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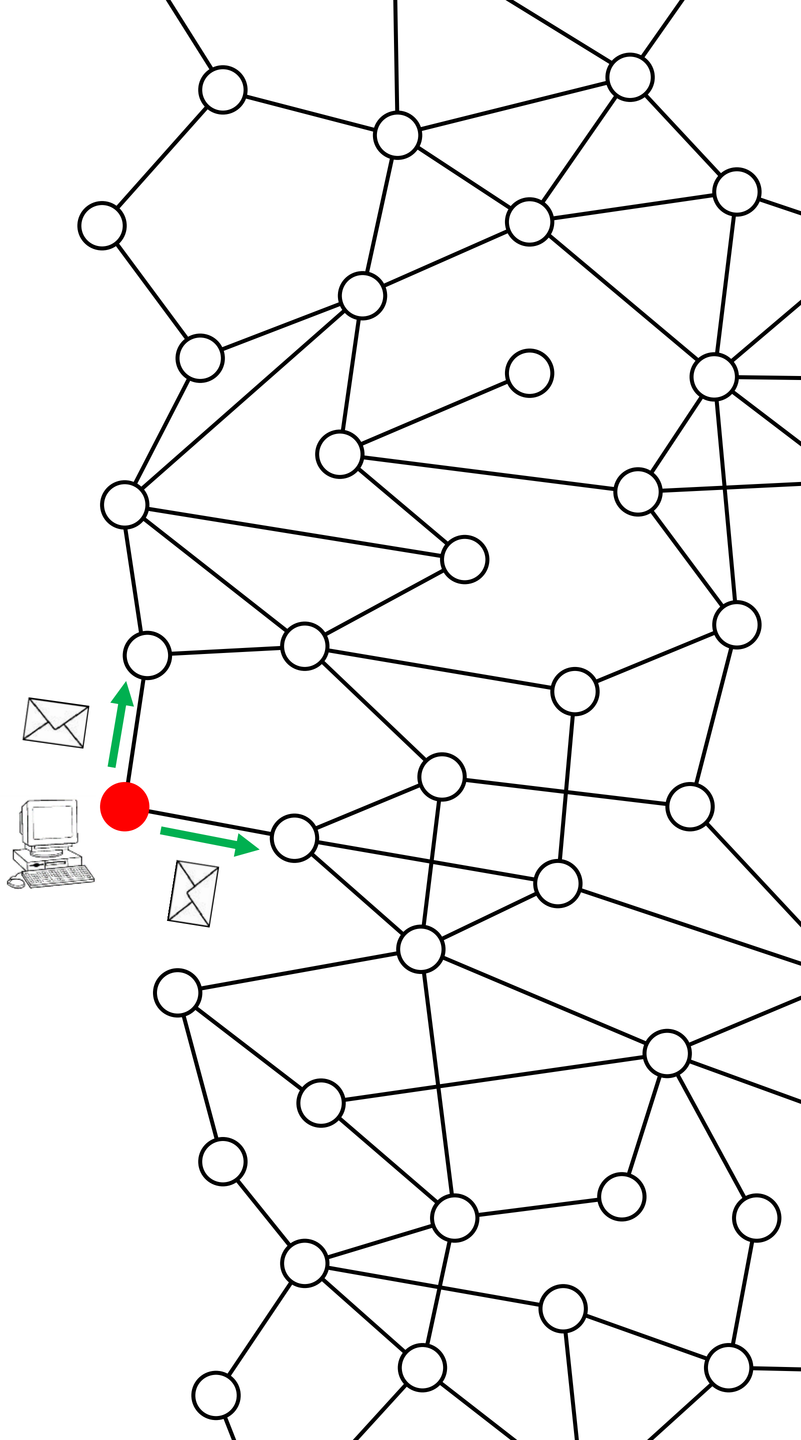

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Synchronous rounds of
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Unlimited message size
and local computation

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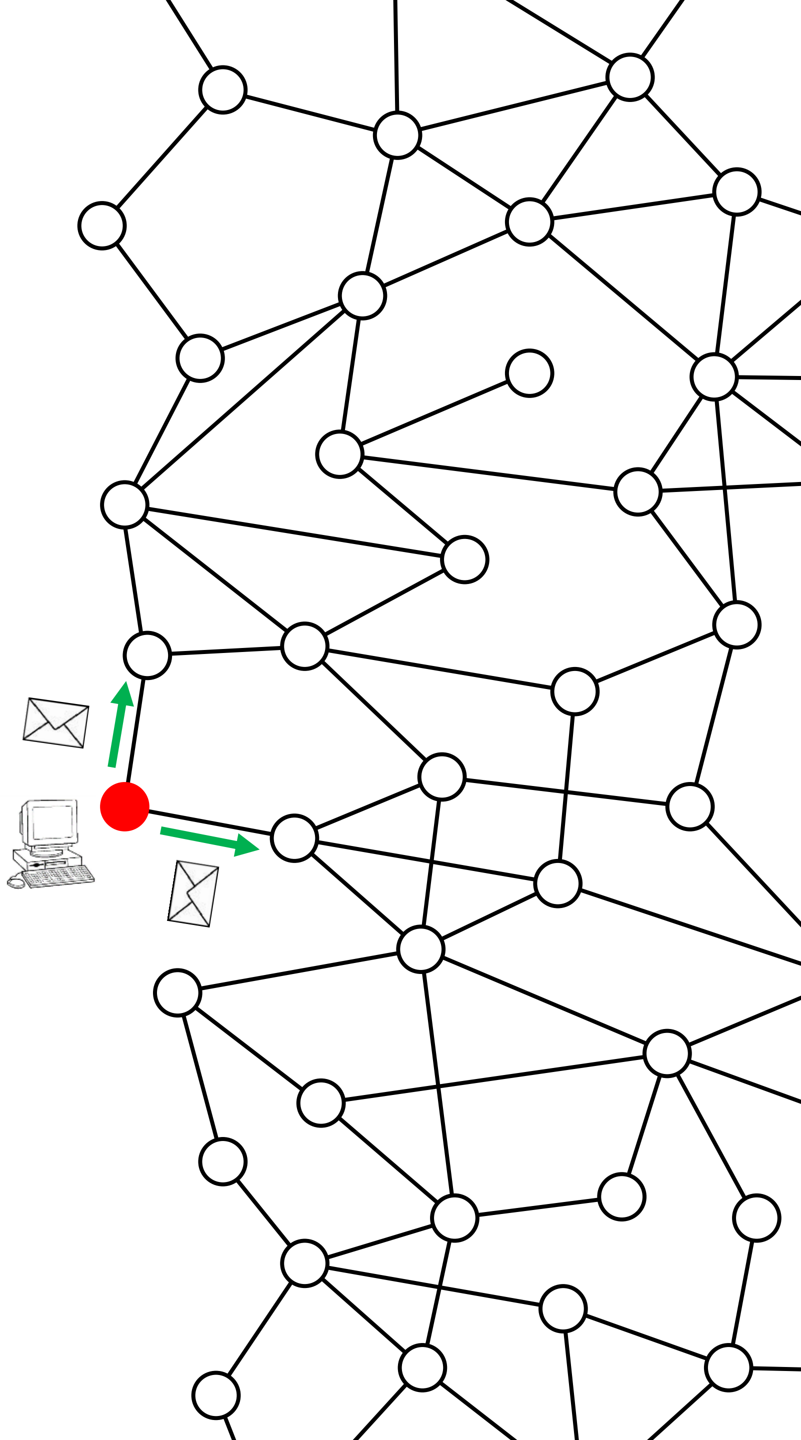

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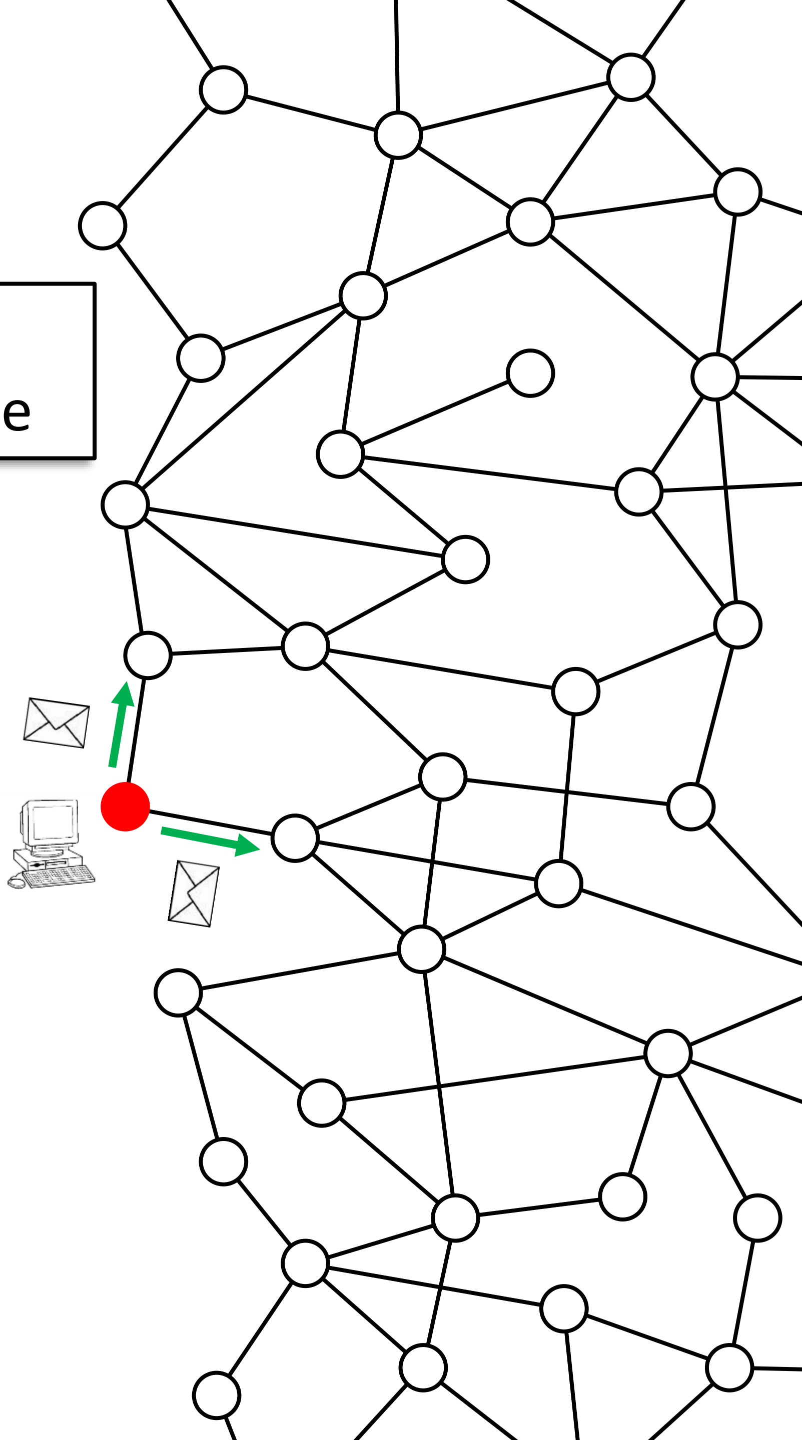

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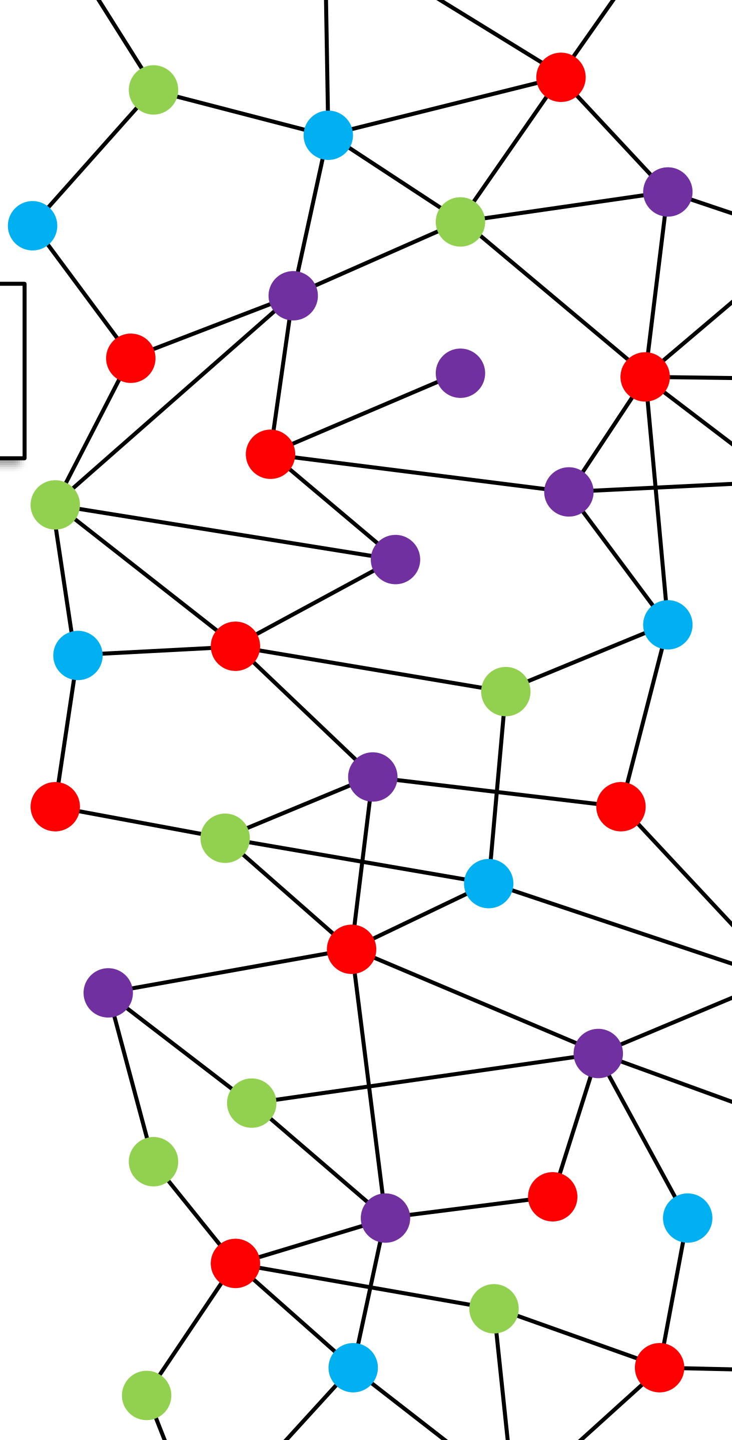

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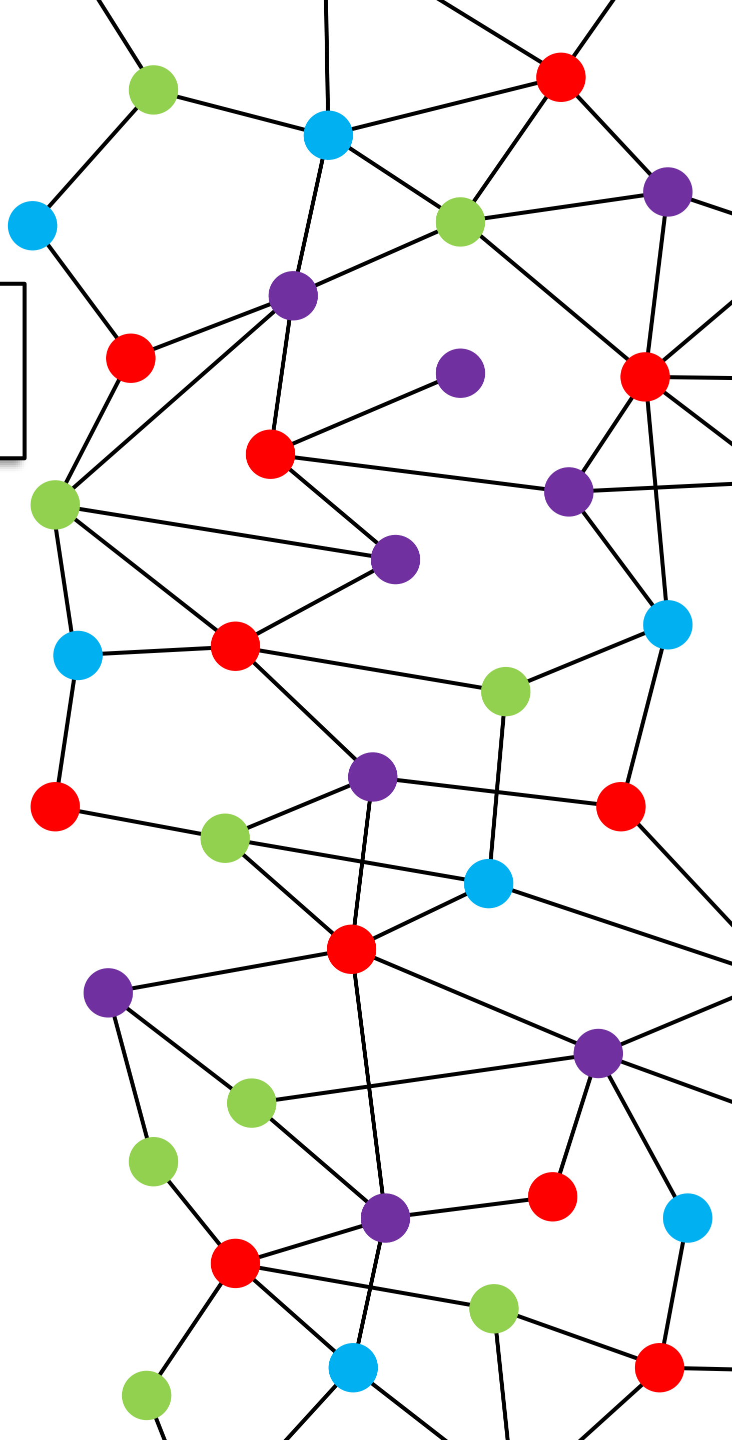

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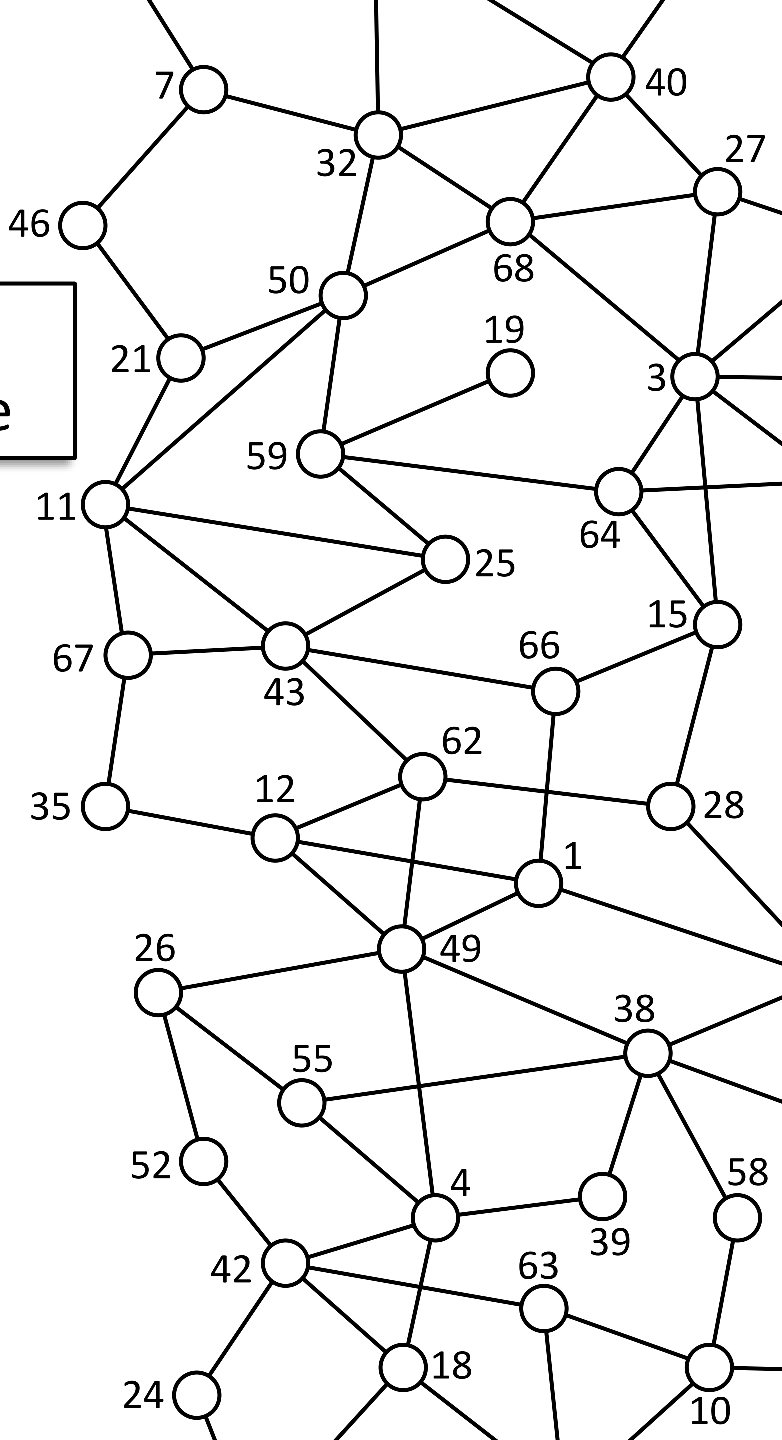
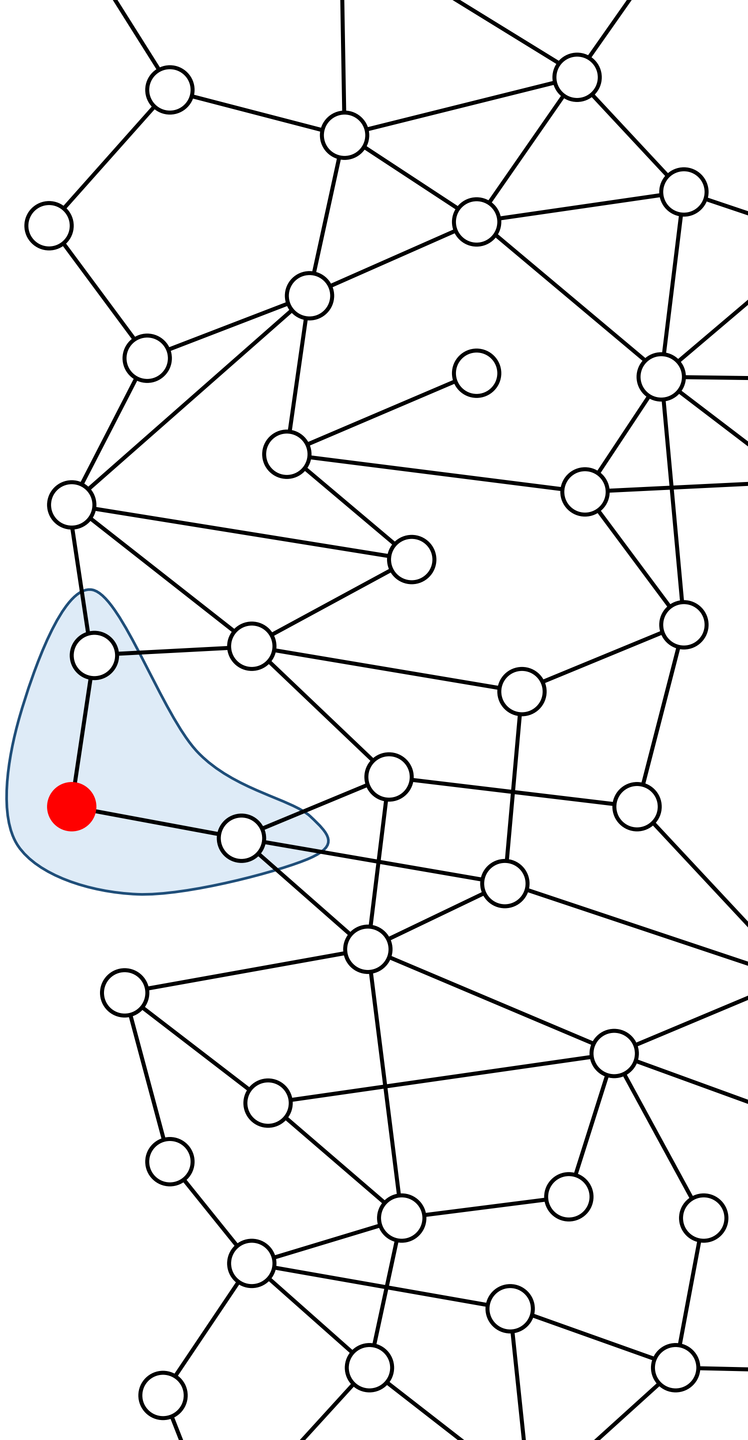


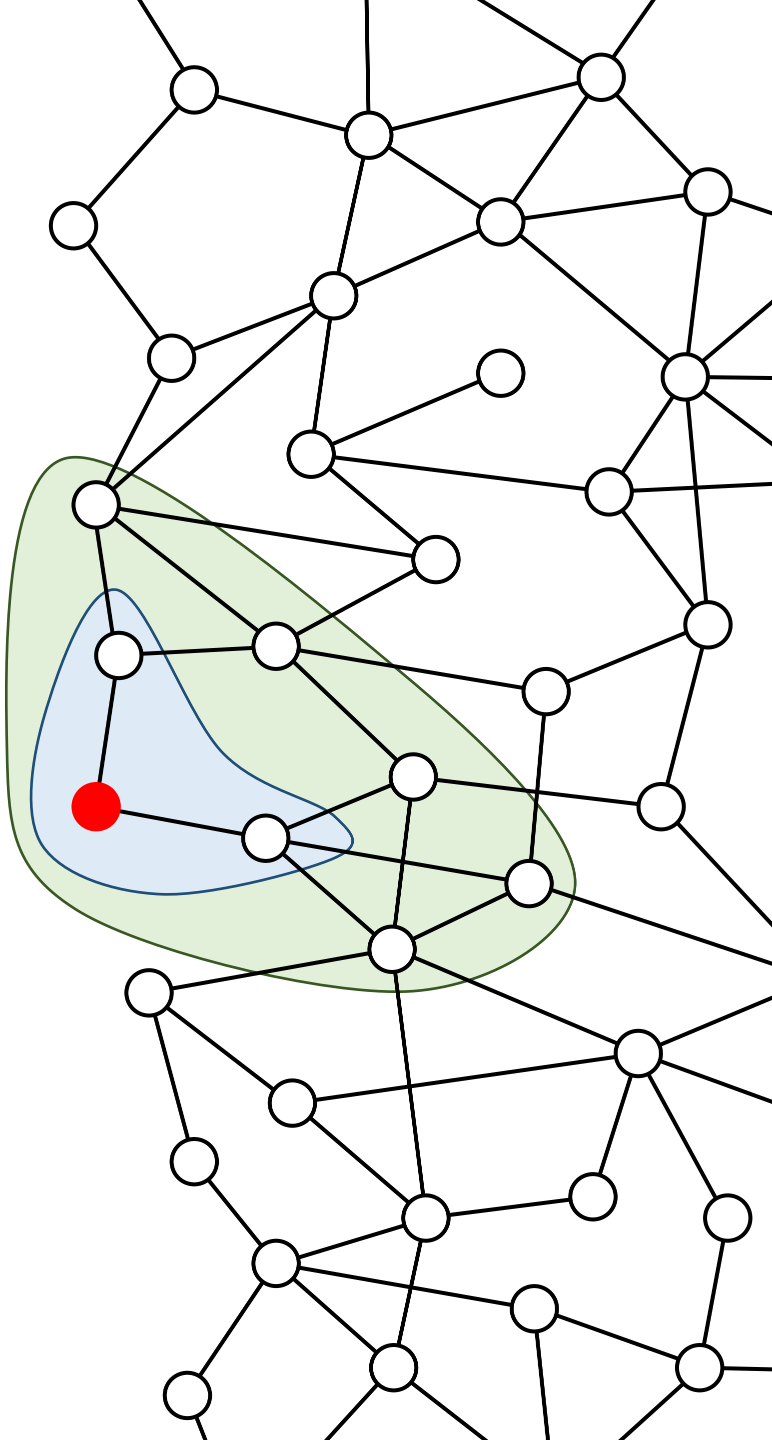
Locality

1 round = distance 1



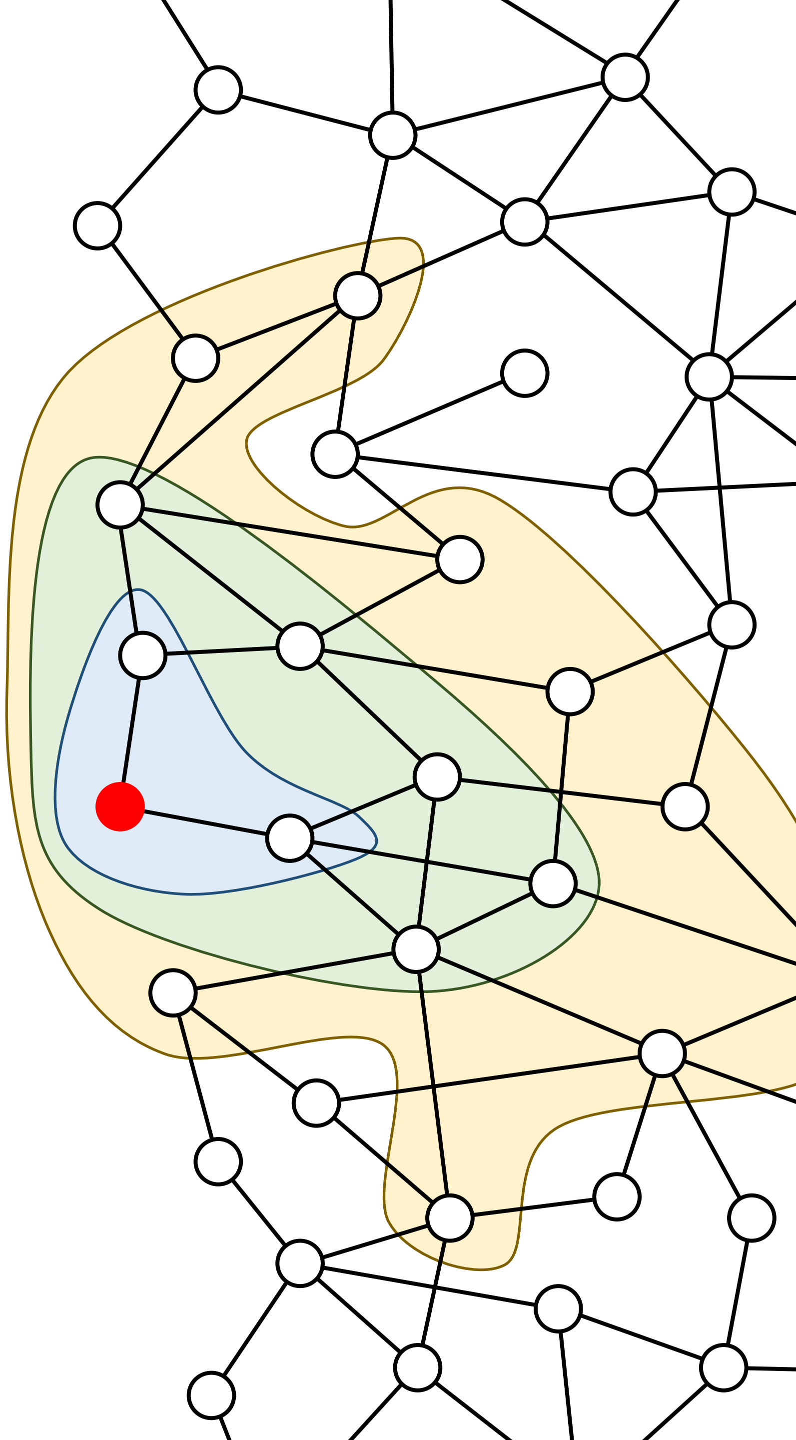
Locality

2 rounds = distance 2



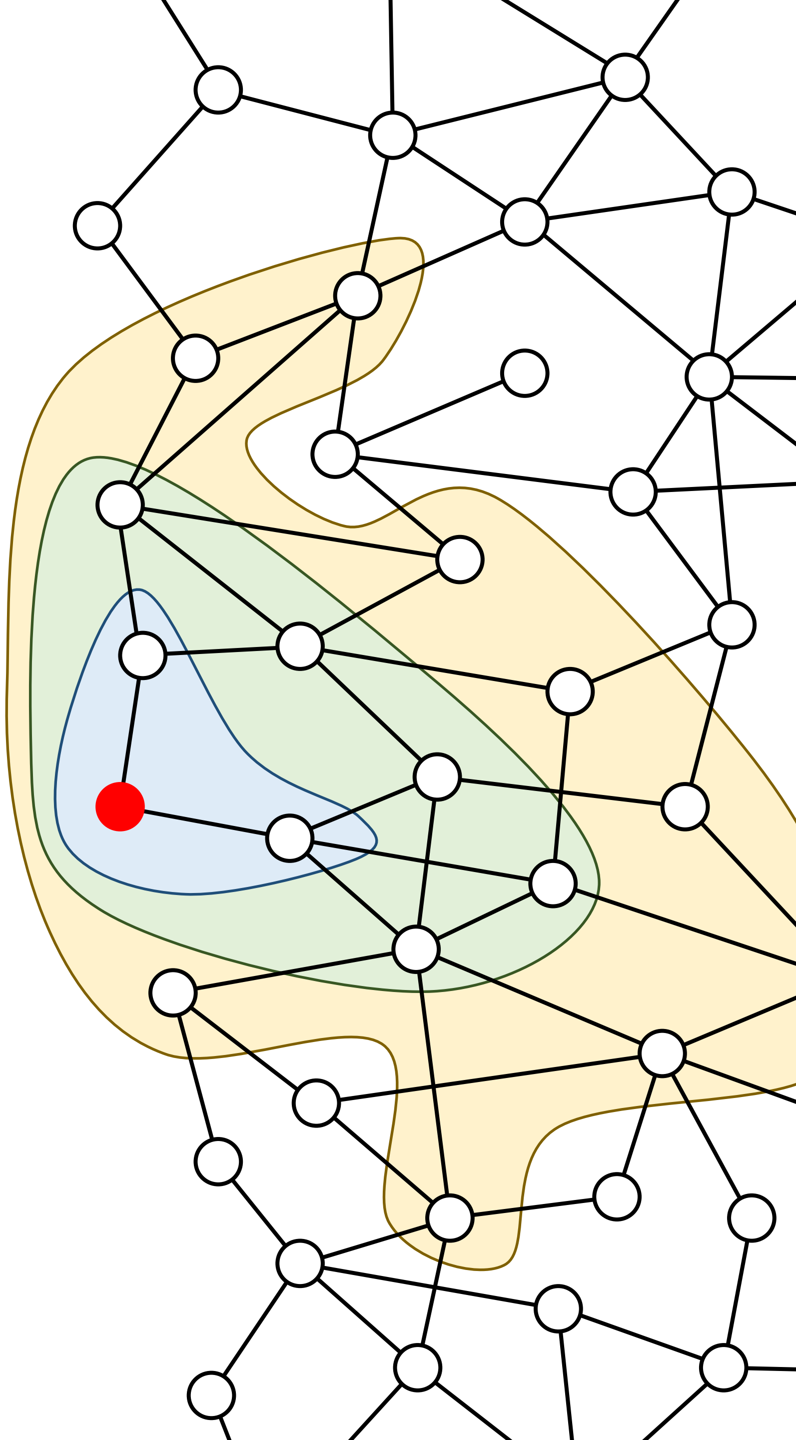
Locality

3 rounds = distance 3



Locality

t rounds $=$ distance t



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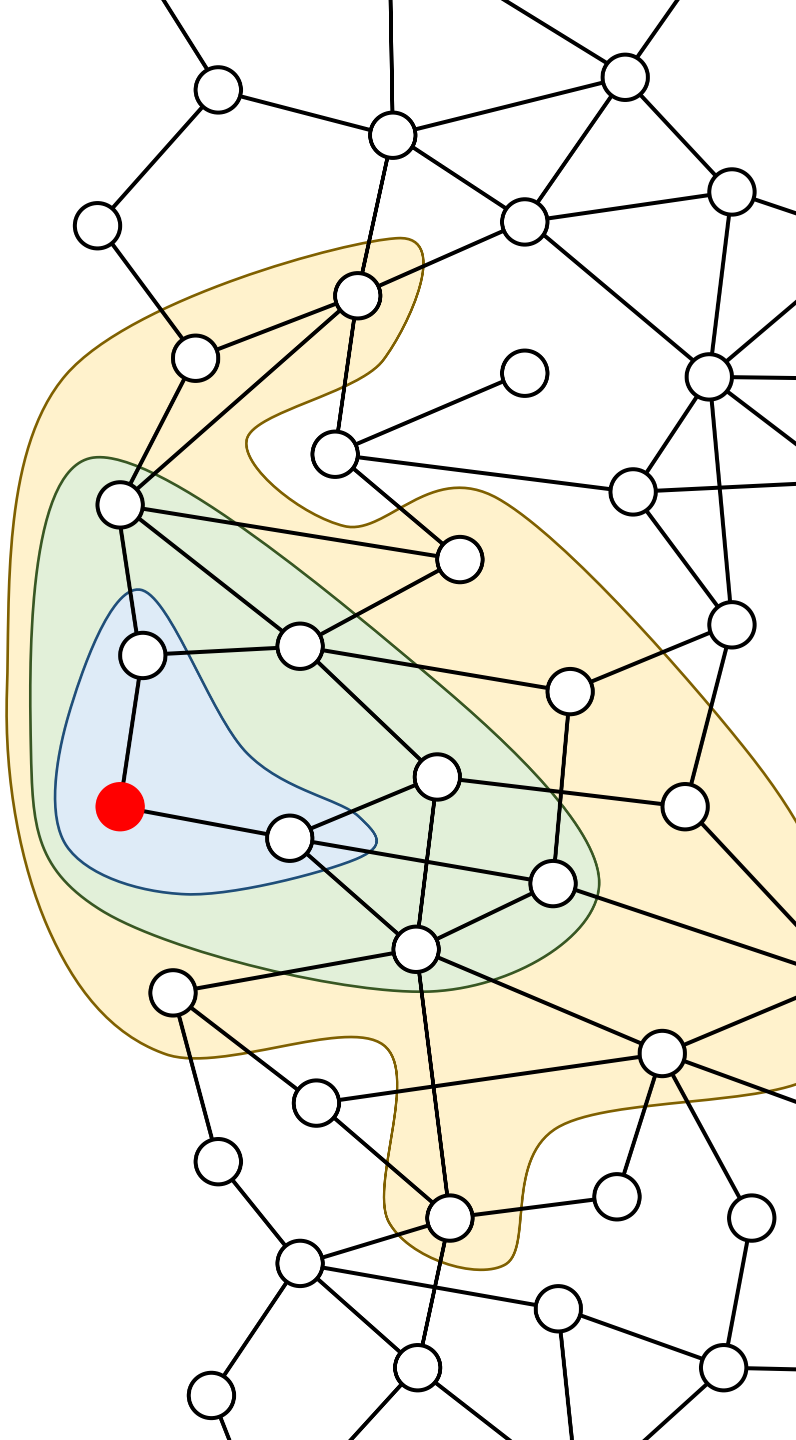

Locally Checkable Problems

Locally Checkable:

- Output correctness is defined via local ($= O(1)$-hop) constraints.

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Sinkless Orientation:

Orient the edges such that no node of degree at least 3 is a sink.

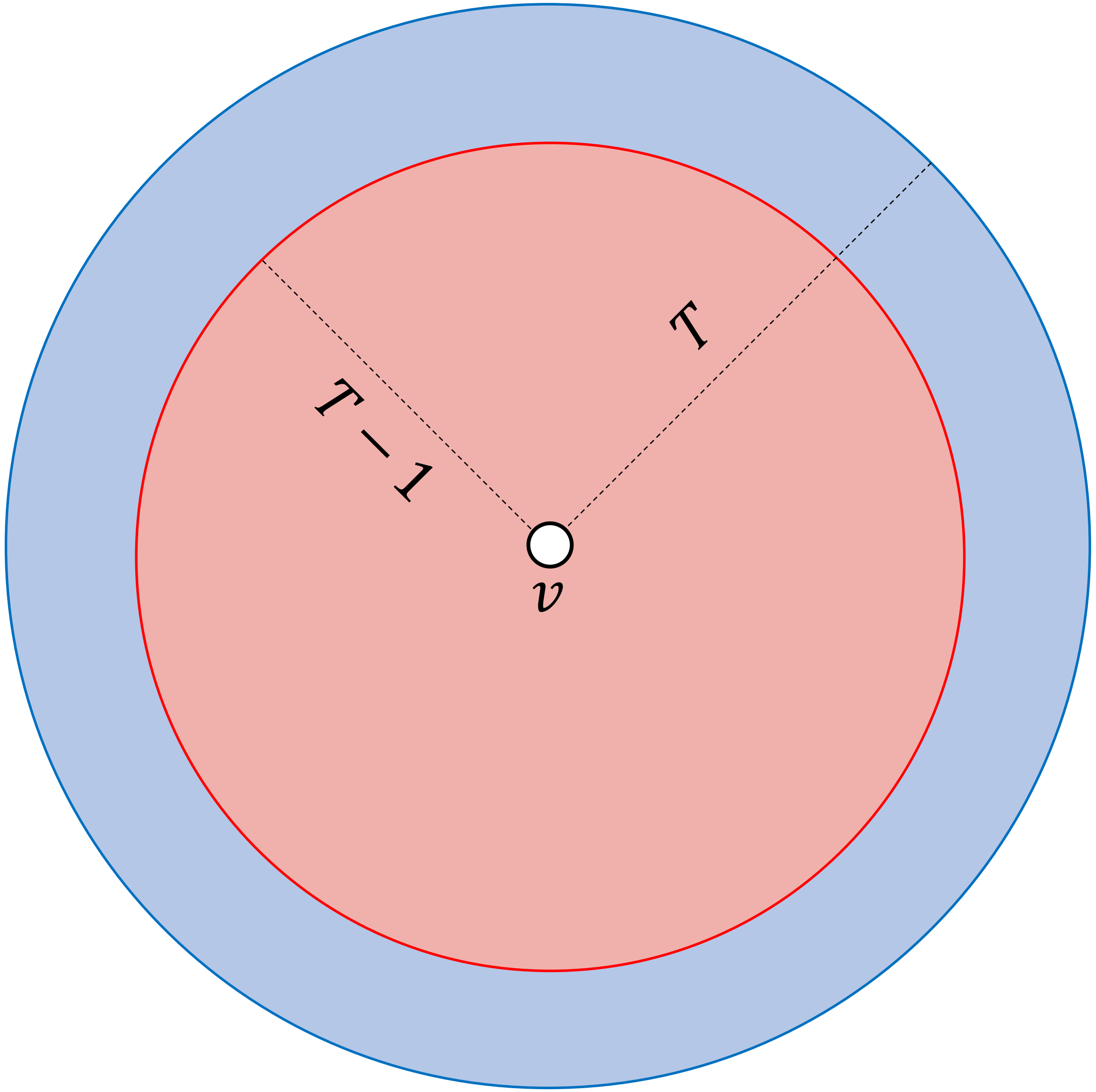
Δ-Coloring:

Compute a proper node coloring with colors $\{1, 2, \dots, \Delta\}$.

Round Elimination

Round Elimination

Given that we can solve some problem in T rounds, what can we do in $T - 1$ rounds?



The Round Elimination Theorem

For any locally checkable problem, we can automatically find a locally checkable problem that is exactly 1 round easier.

[Brandt, 2019]

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high-girth graphs

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Round Elimination

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \quad \dots$$

$$T_0 \qquad T_1 = T_0 - 1 \qquad T_2 = T_1 - 1$$

Round Elimination

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \quad \dots \quad \xrightarrow{\mathcal{R}} \Pi_k$$

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If, for all $i \geq 0$, Π_i is not 0-round solvable ("non-trivial"), then solving Π_0 takes $\Omega(\log n)$ rounds in the LOCAL model.

Fixed Points

Fixed point: problem Π satisfying $\mathcal{R}(\Pi) = \Pi$

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Π non-trivial fixed point $\Longrightarrow$ lower bound of $\Omega(\log n)$ for Π

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Relaxation of Π: problem Π' that can be solved in 0 rounds given a solution to Π

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Is every problem of complexity $\Omega(\log n)$ relaxable to a non-trivial fixed point?

Π non-trivial fixed point $\Longrightarrow$ lower bound of $\Omega(\log n)$ for Π

relaxable to a

$\overset{?}{\Longleftarrow}$

Any algorithm solving Sinkless Orientation requires $\Omega(\log n)$ rounds.

Marks' Technique

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$(\Delta \in O(1))$

Marks' Technique

assume that there is a
$o(\log n)$-round algorithm $\mathcal{A}$

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define a set of two-player
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show: each possible distribution
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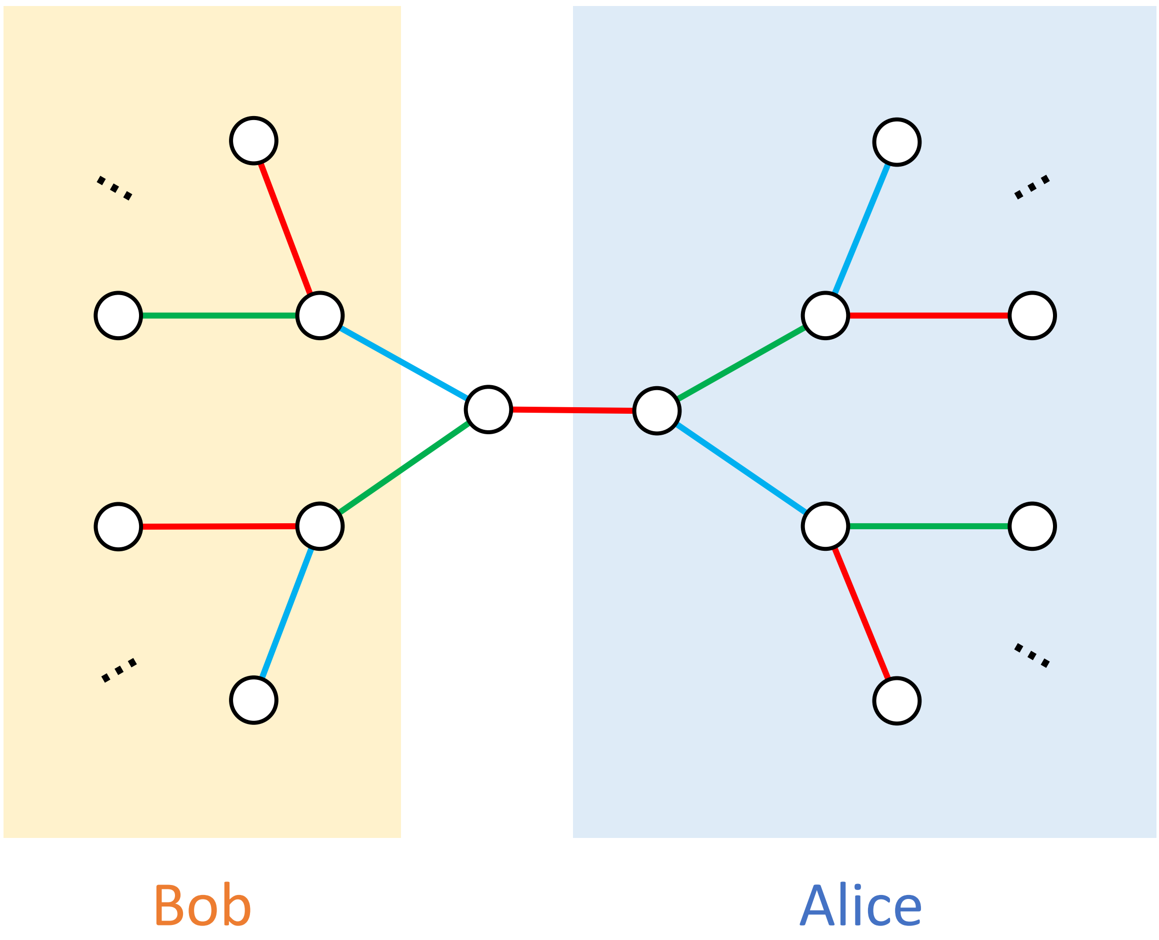

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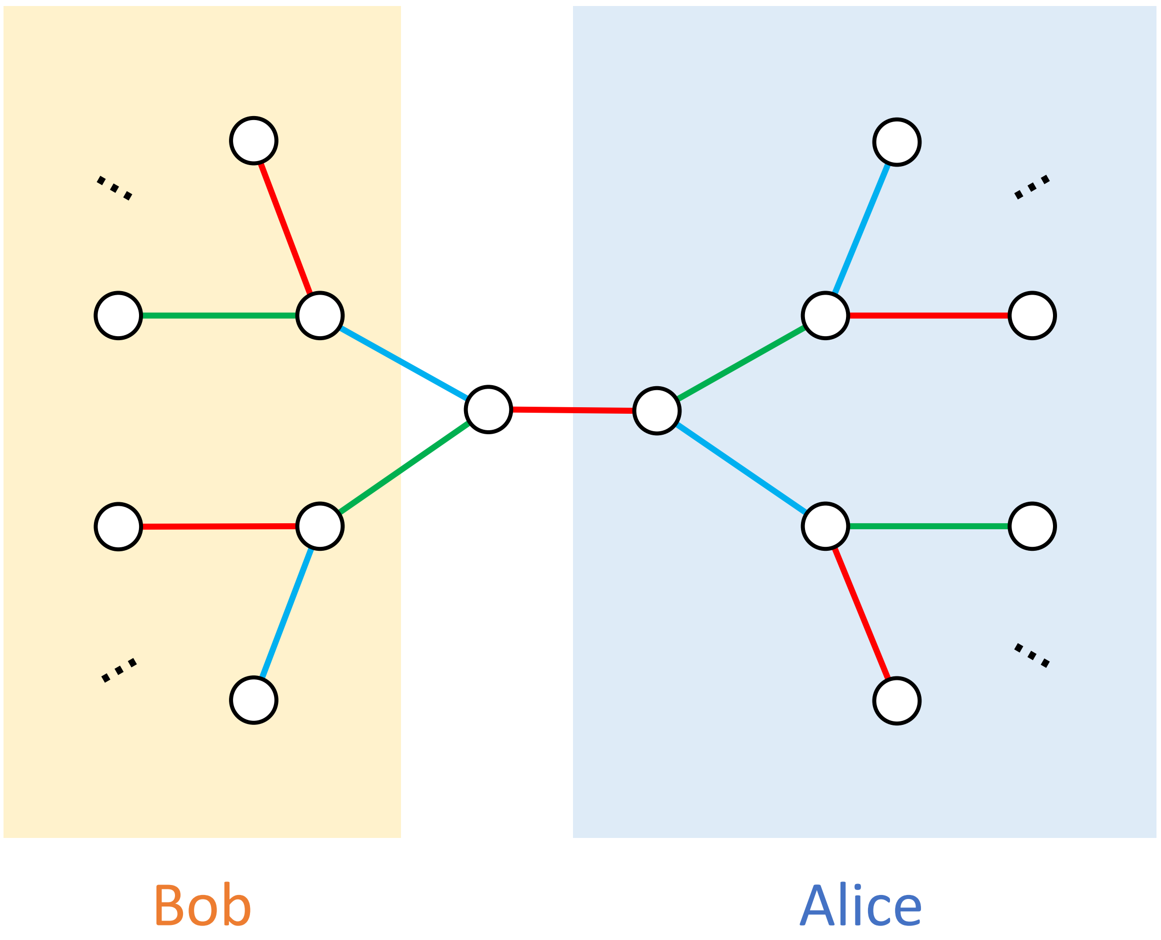

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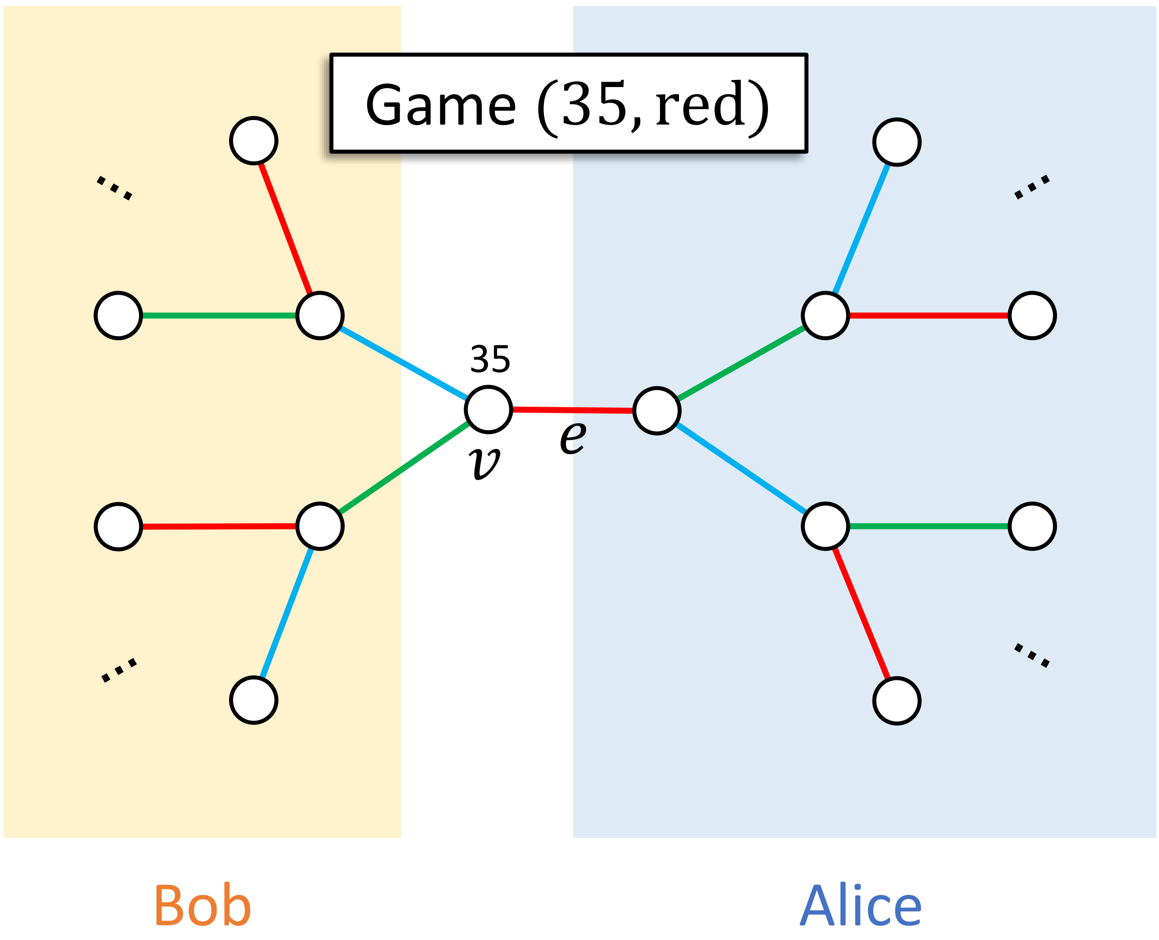

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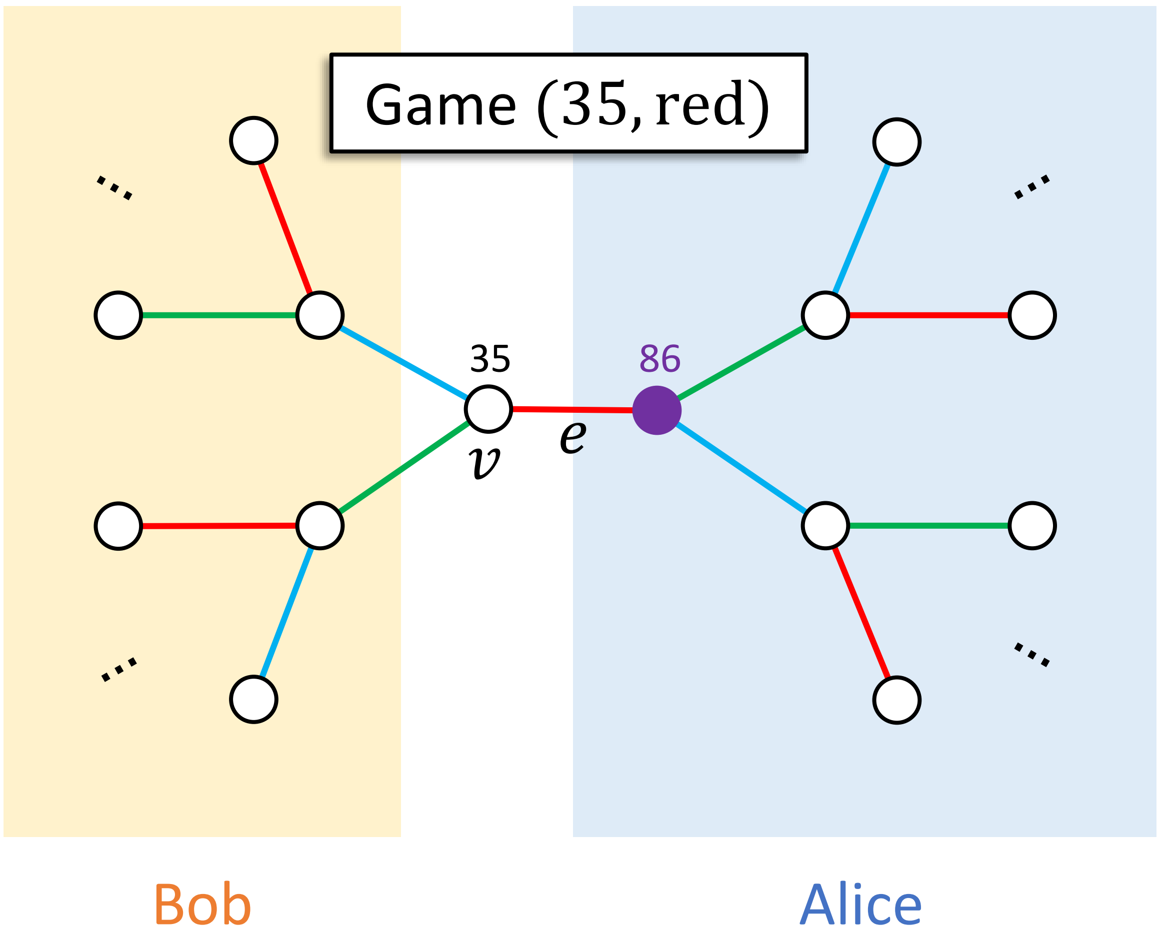

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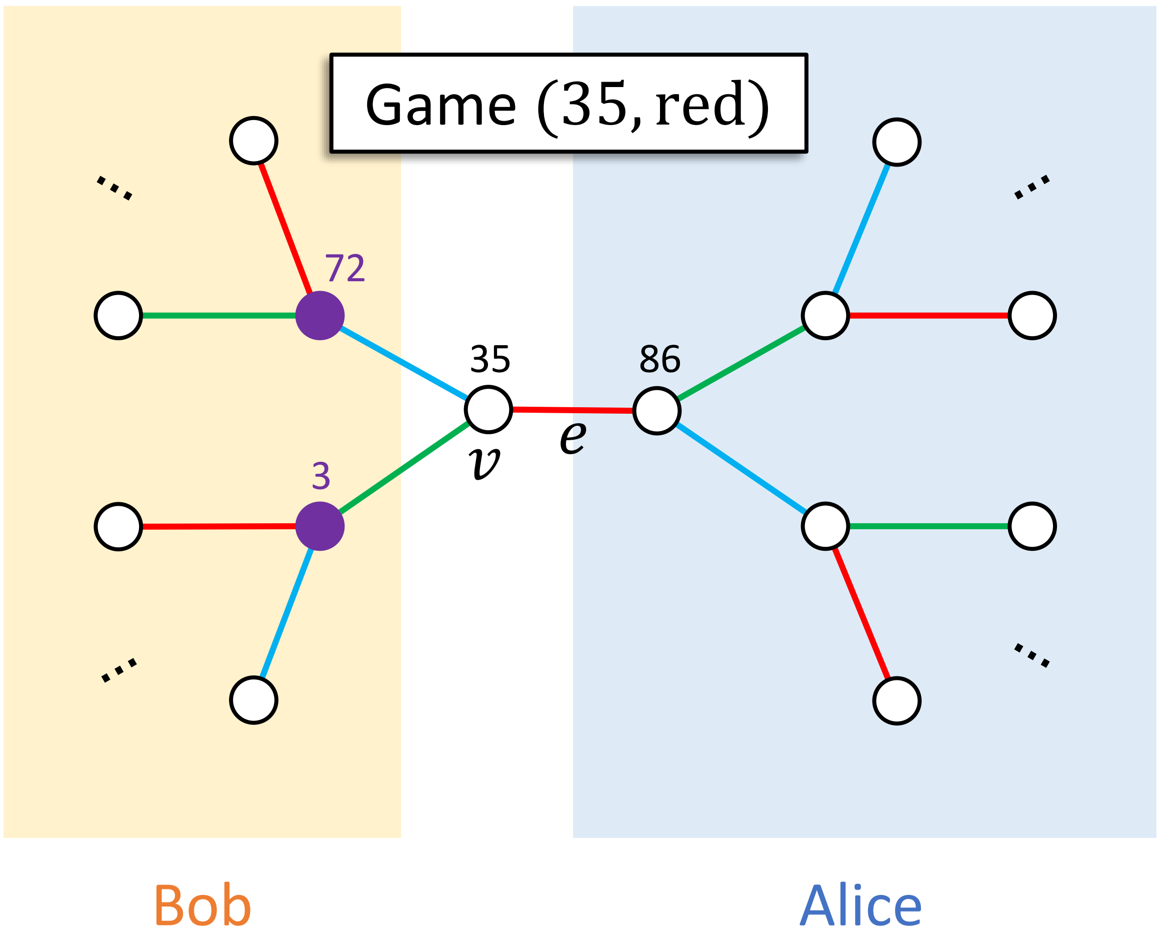

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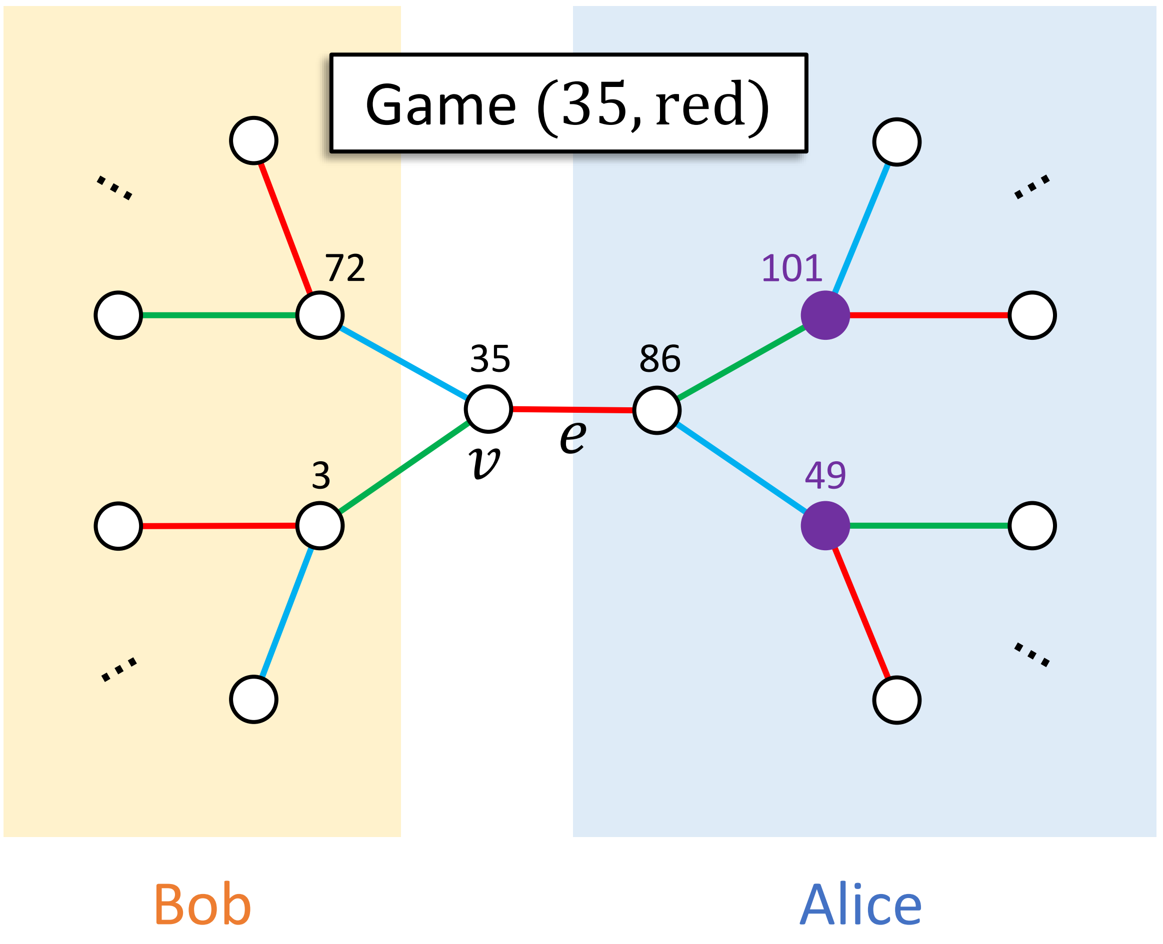

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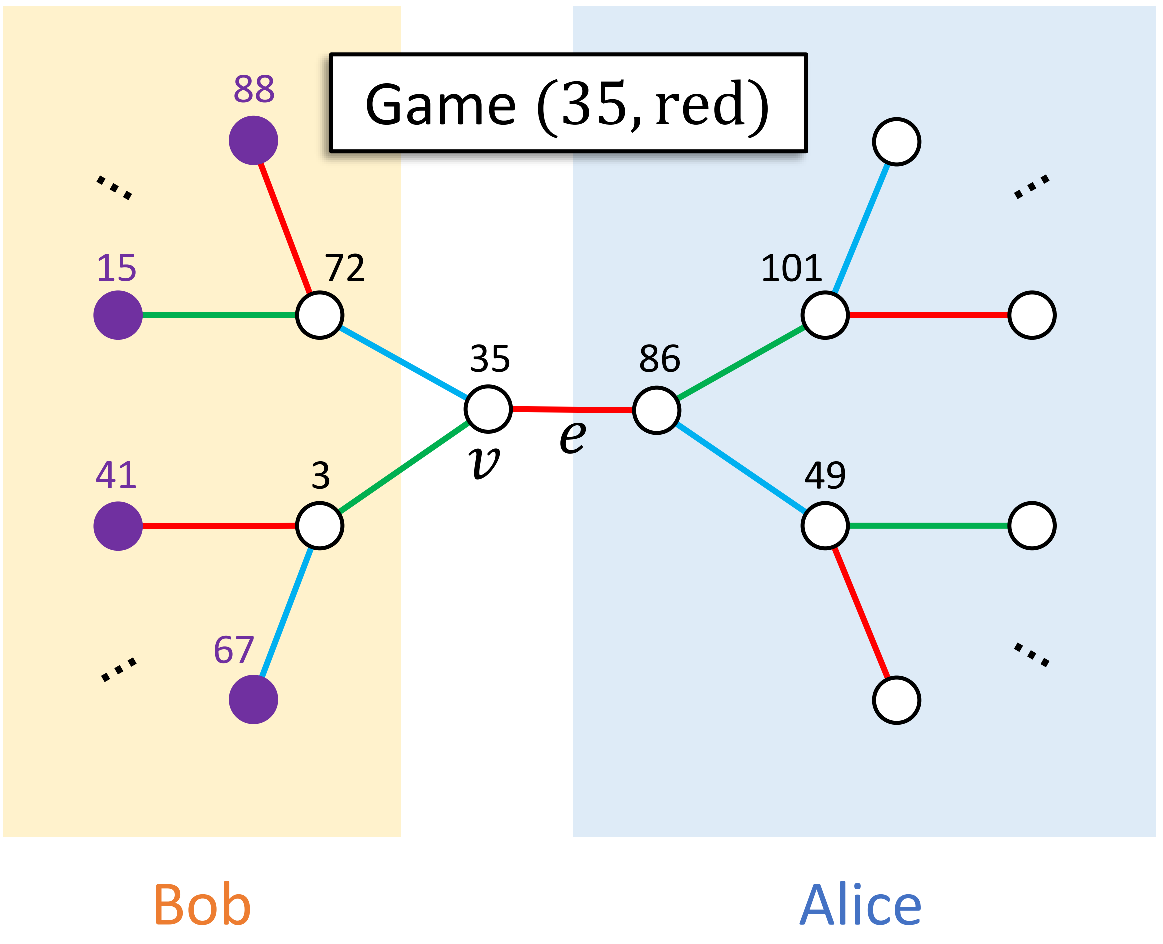

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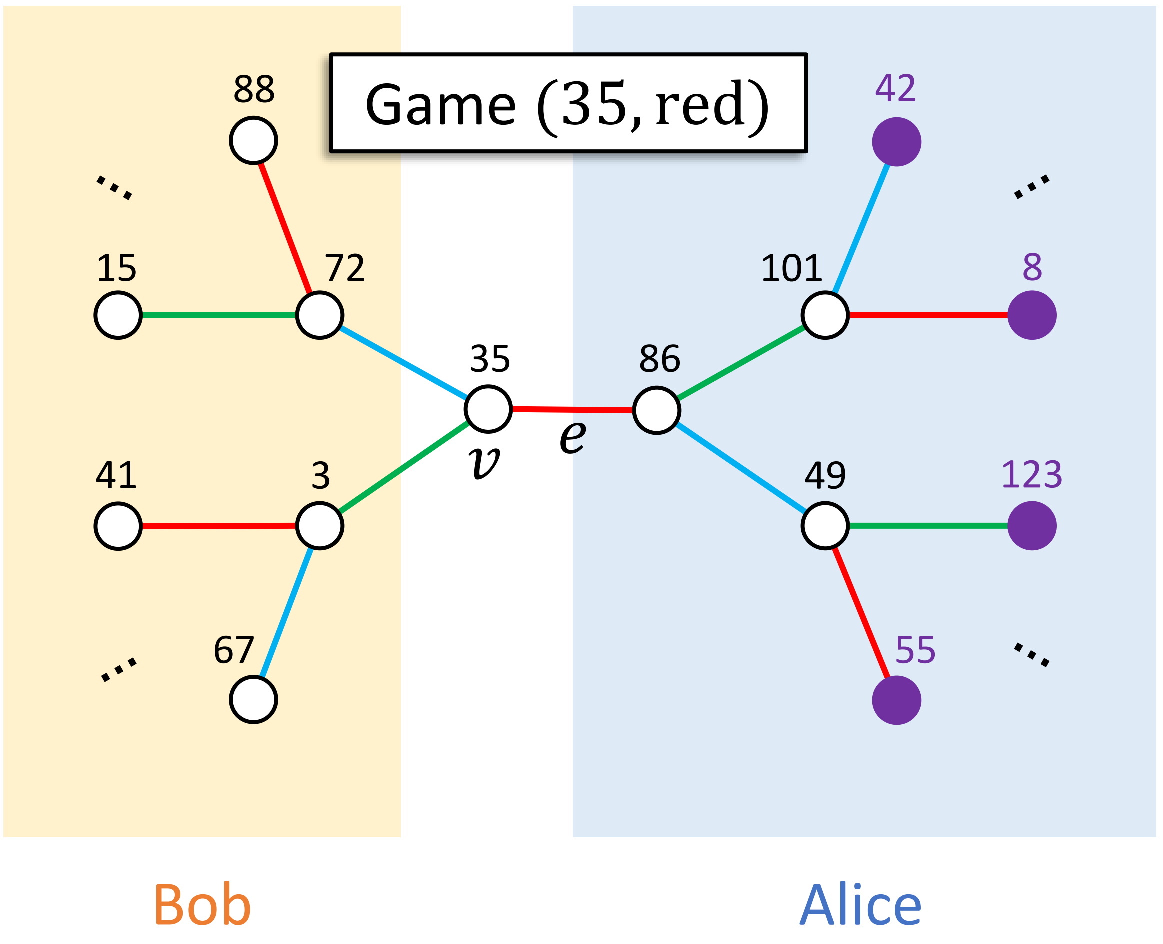

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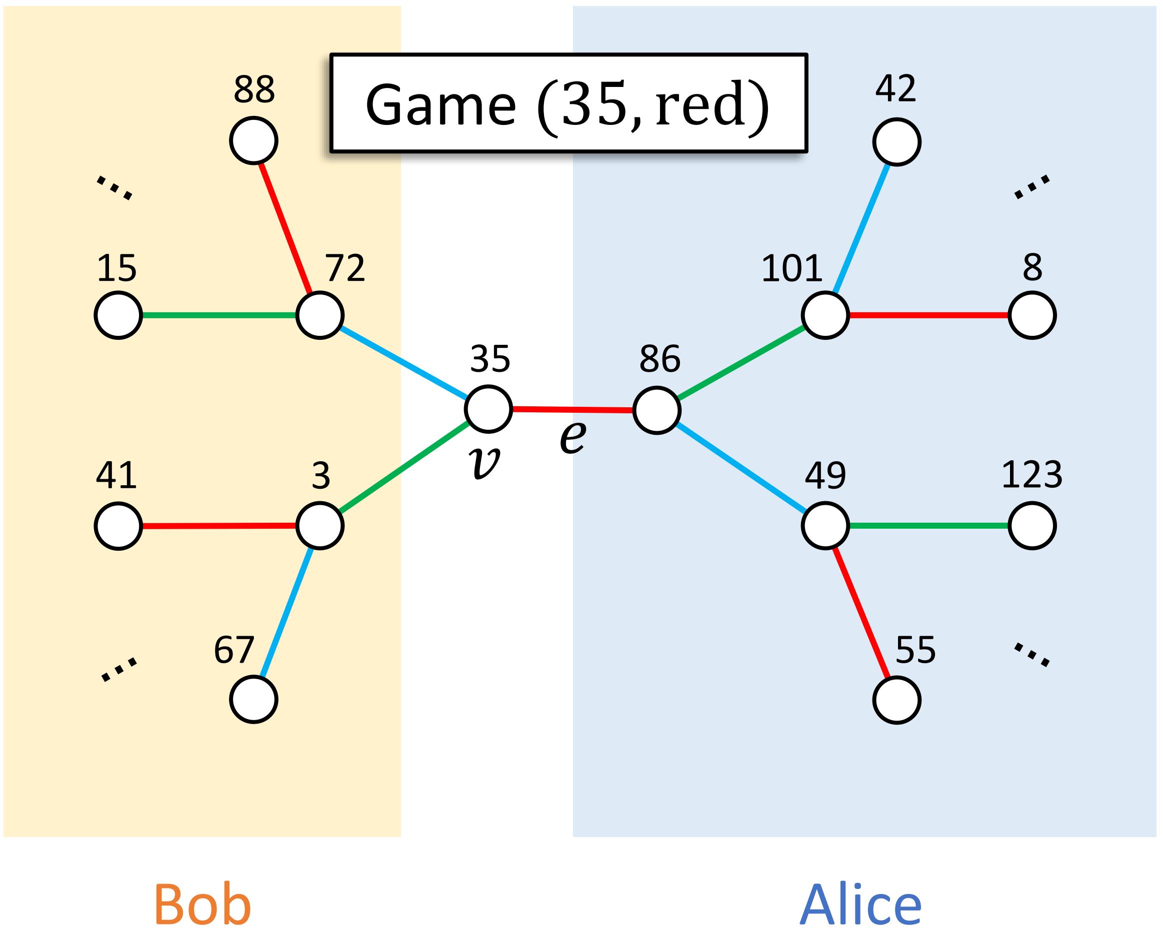

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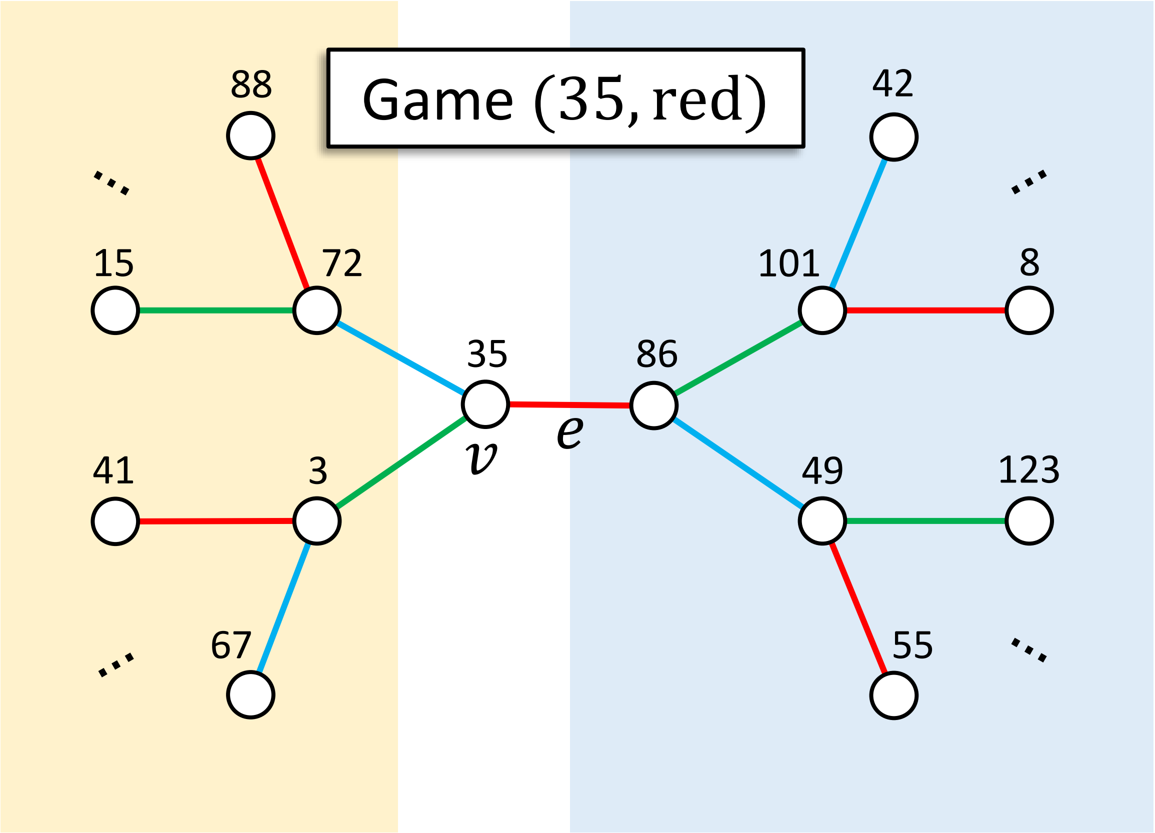

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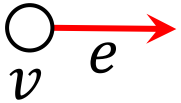
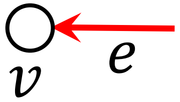


winning condition



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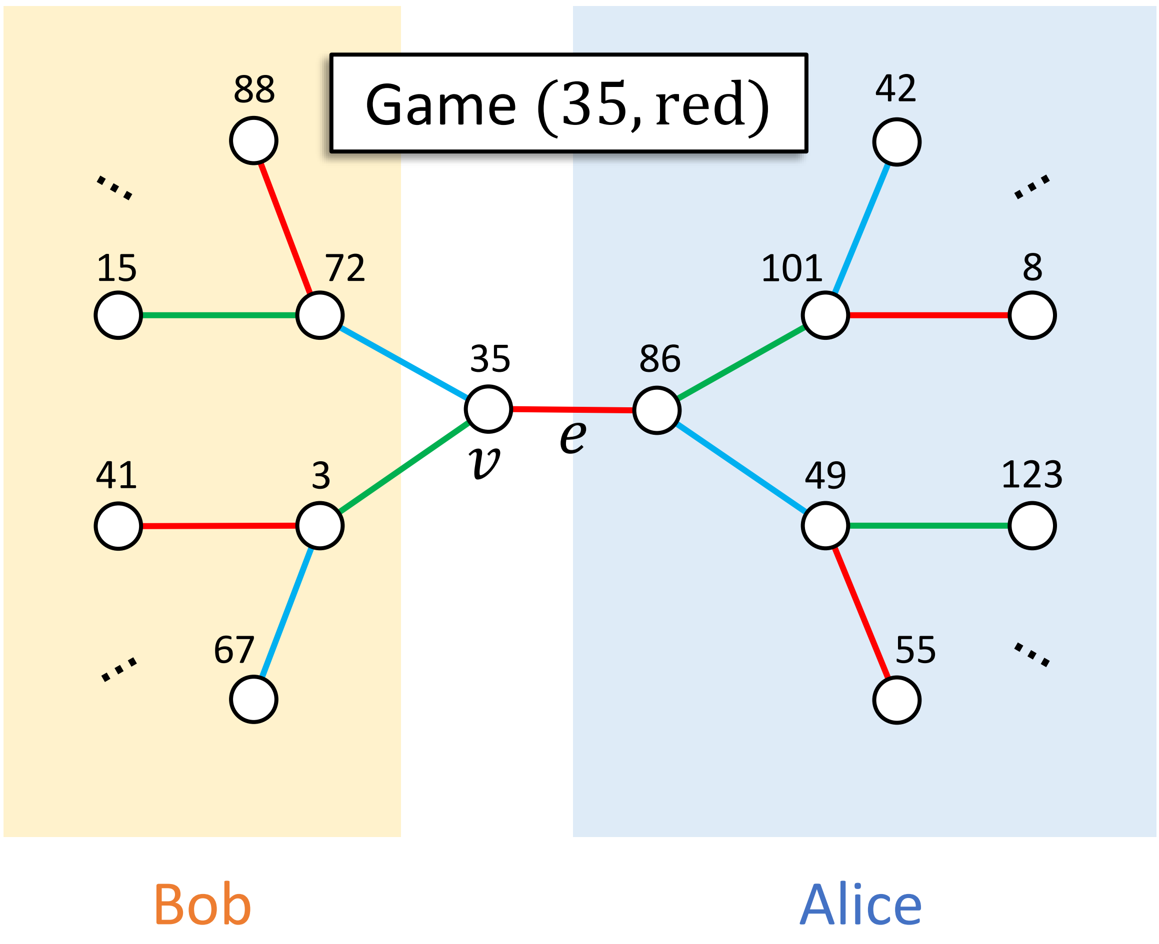

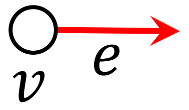

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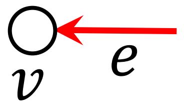

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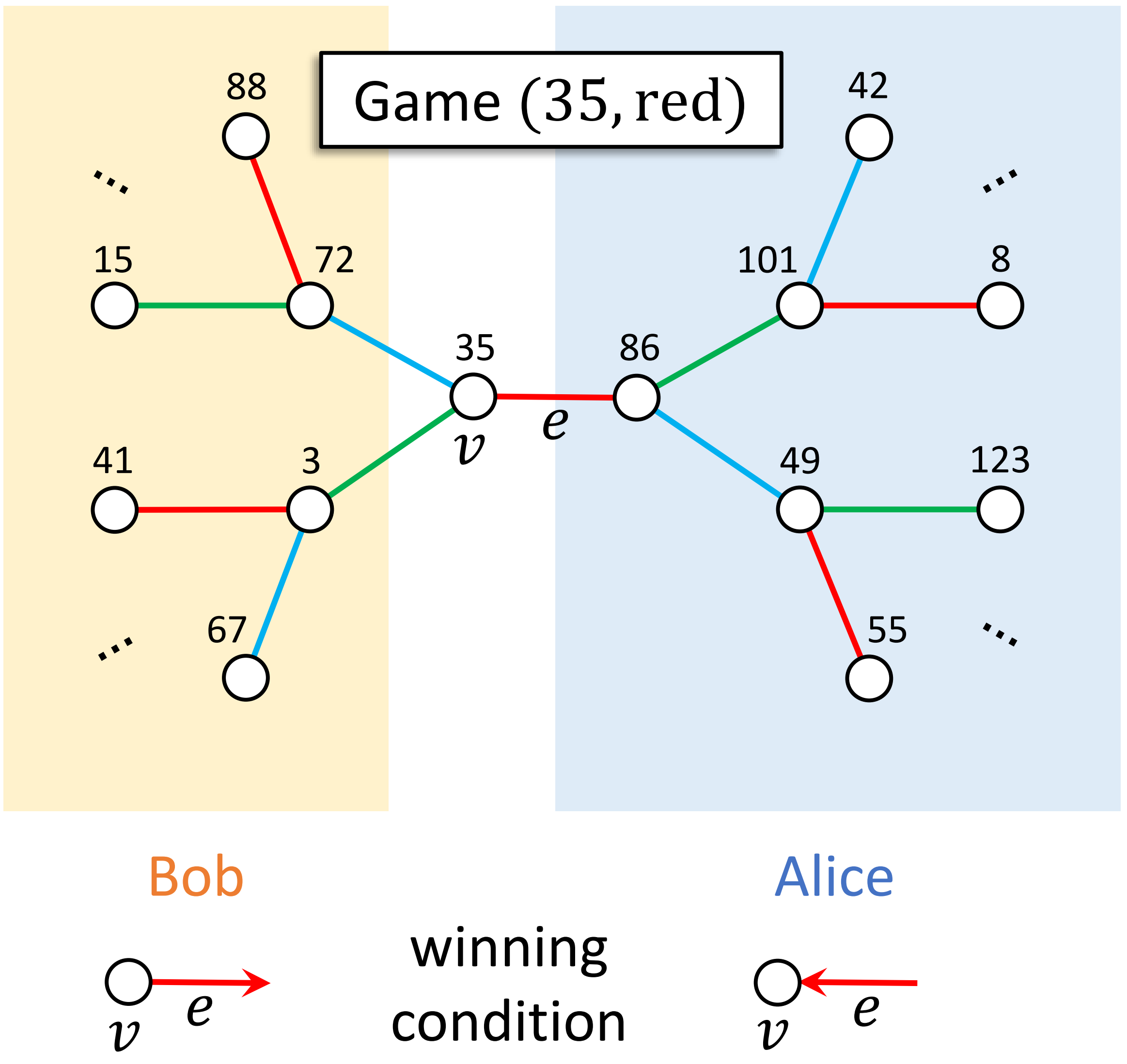

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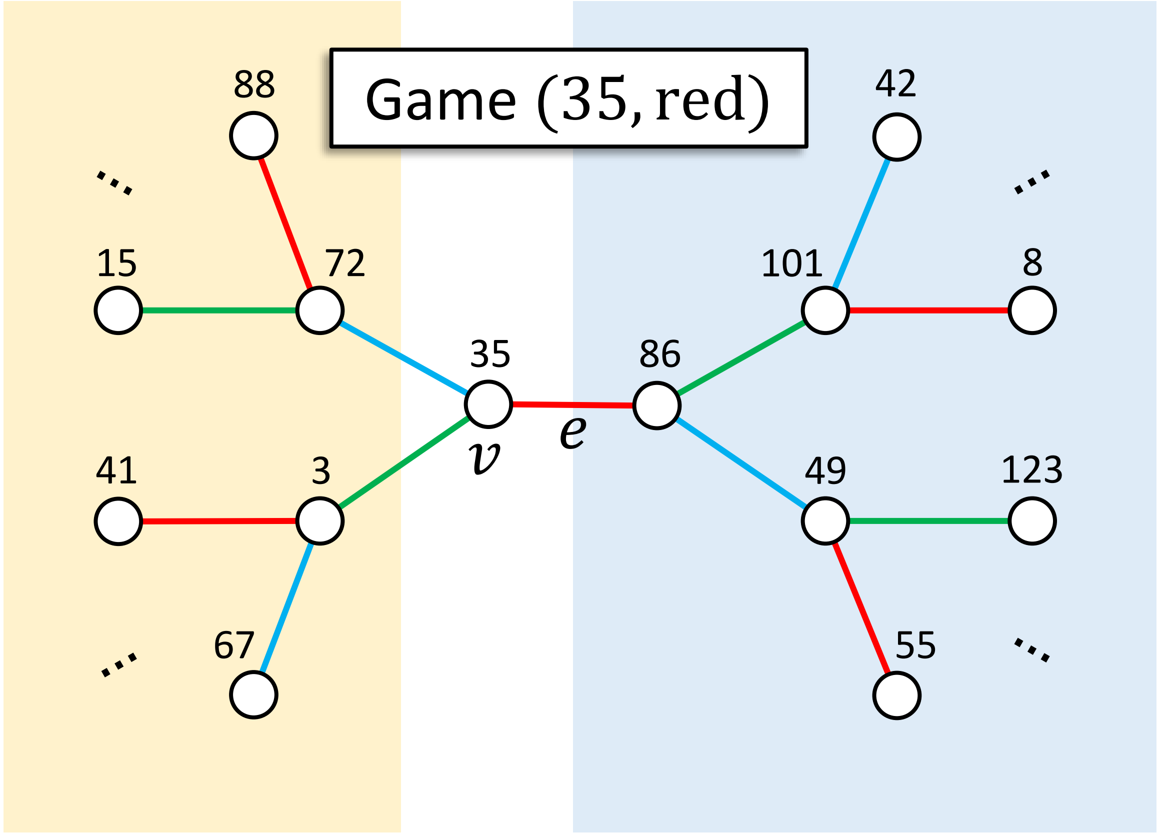

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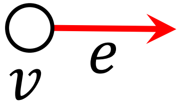

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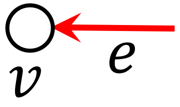

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Let's play Alice's strategies against each other!

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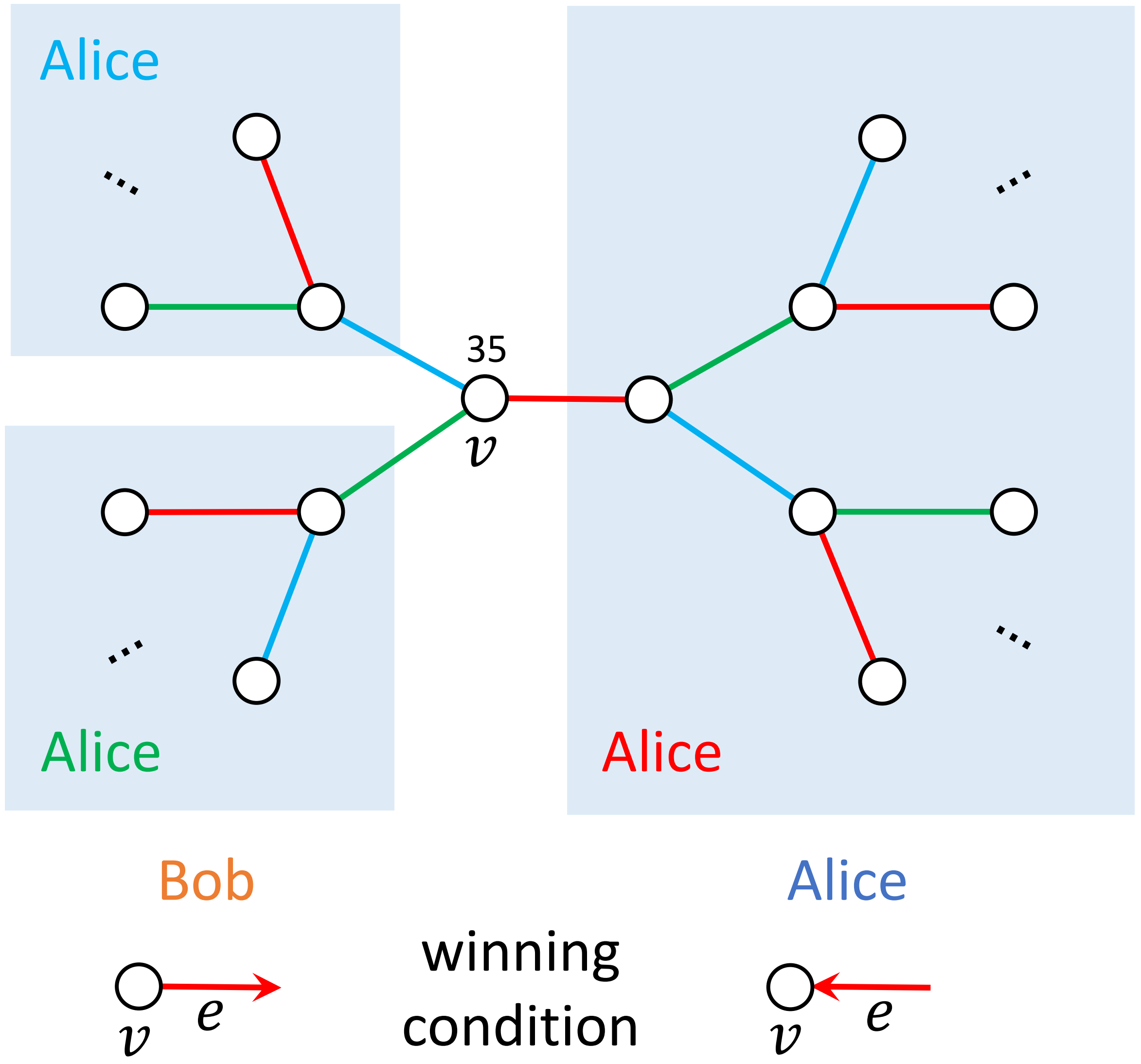

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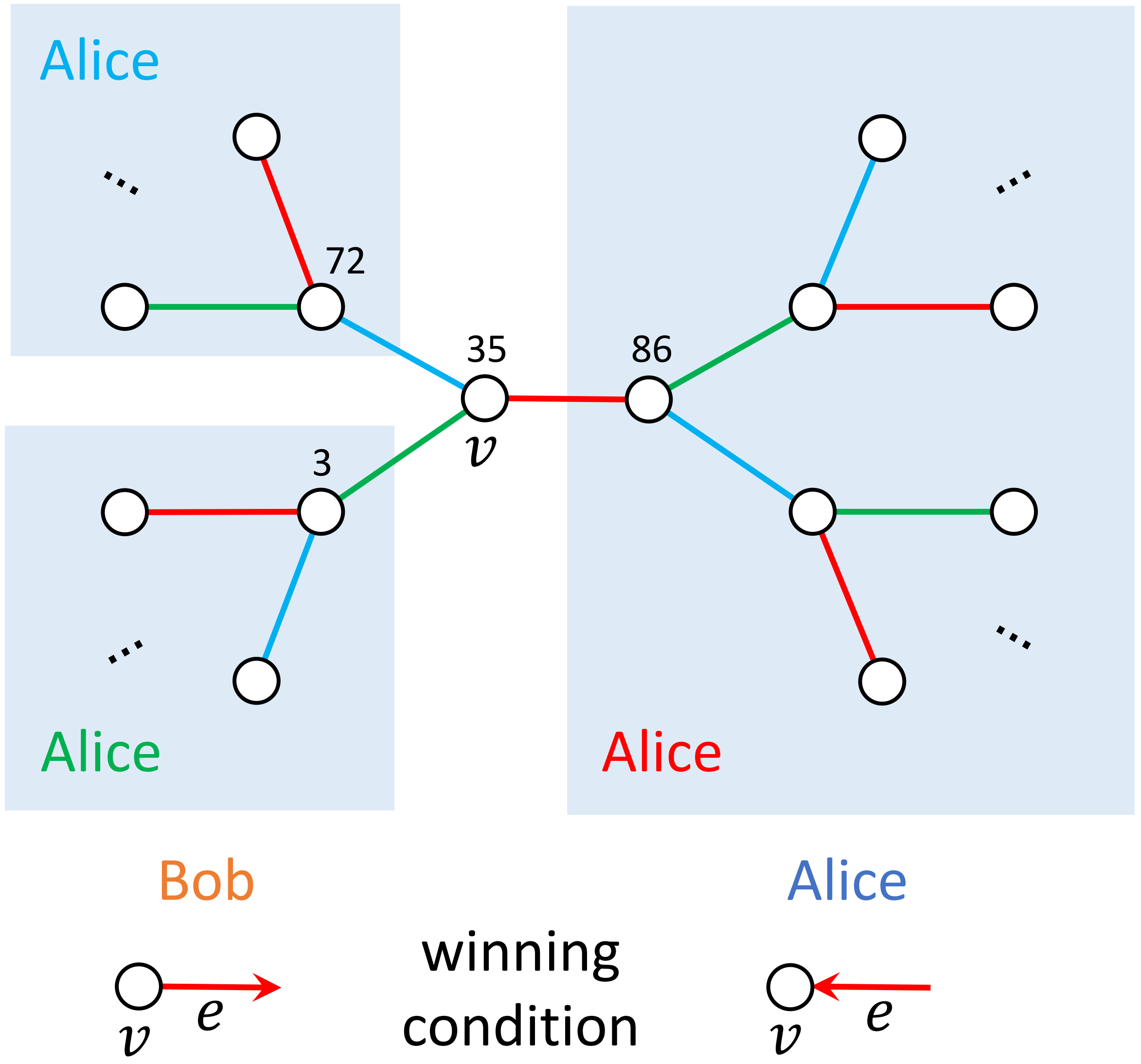

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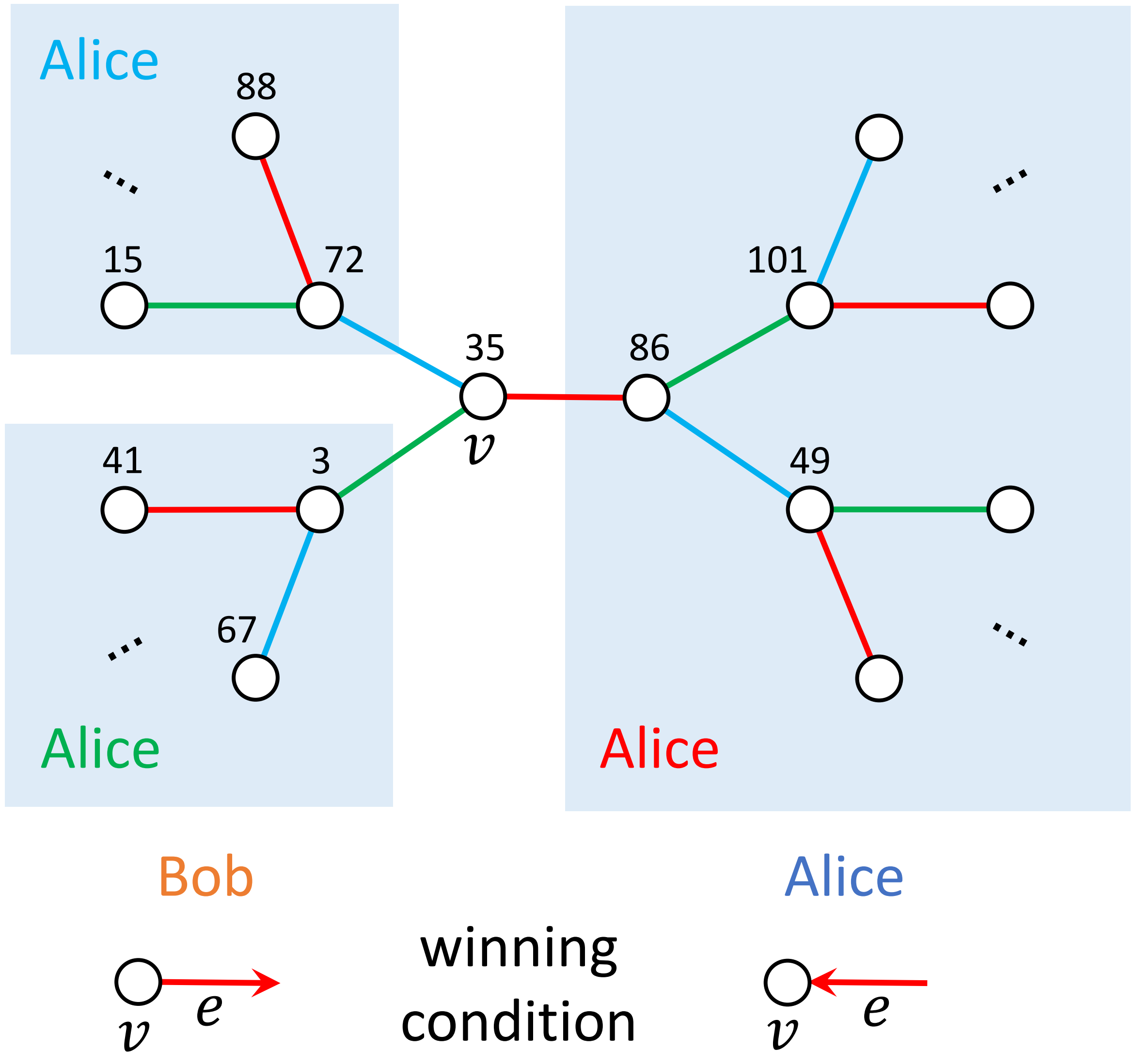

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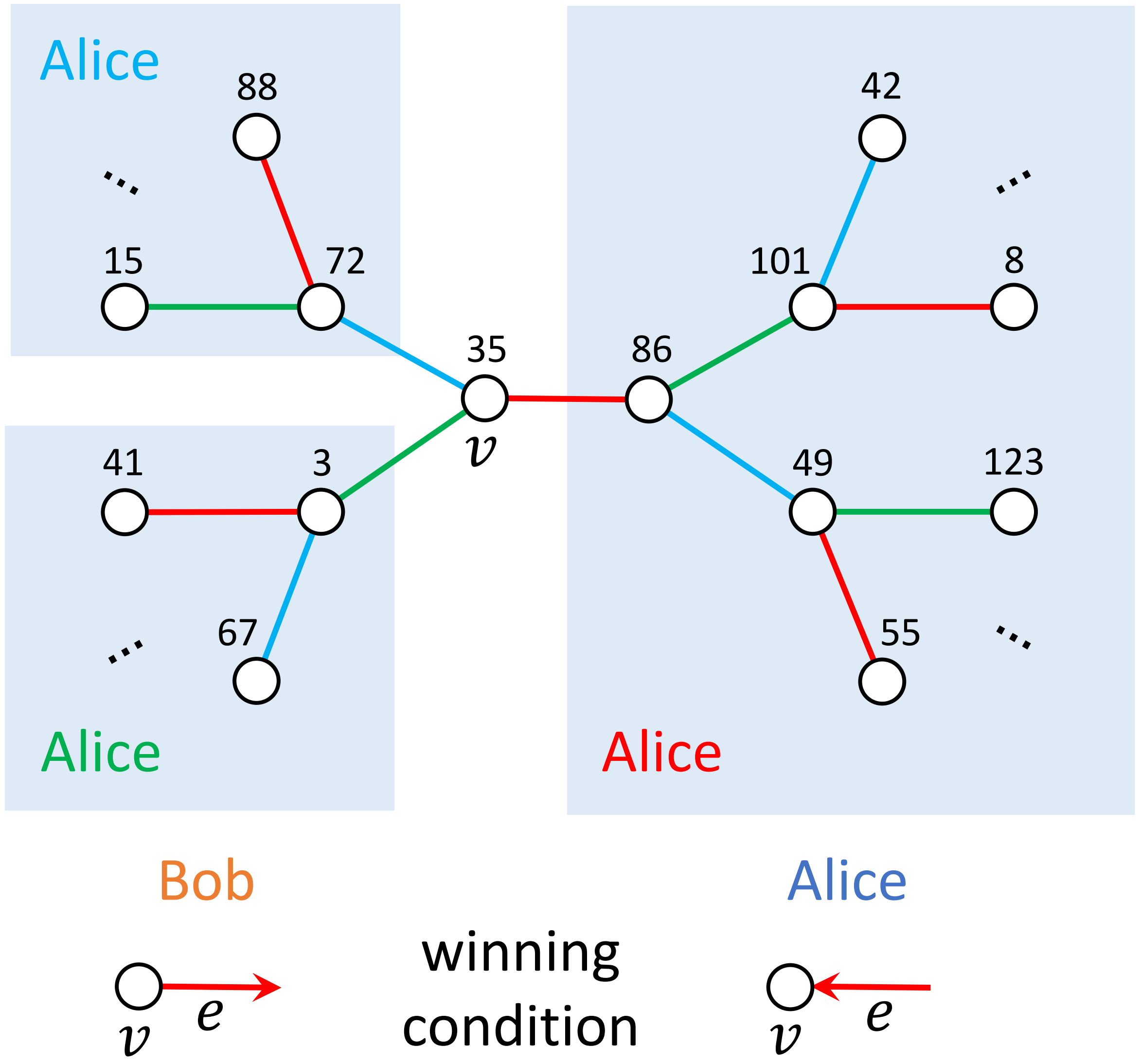

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define a set of two-player games based on $\mathcal{A}$

↓

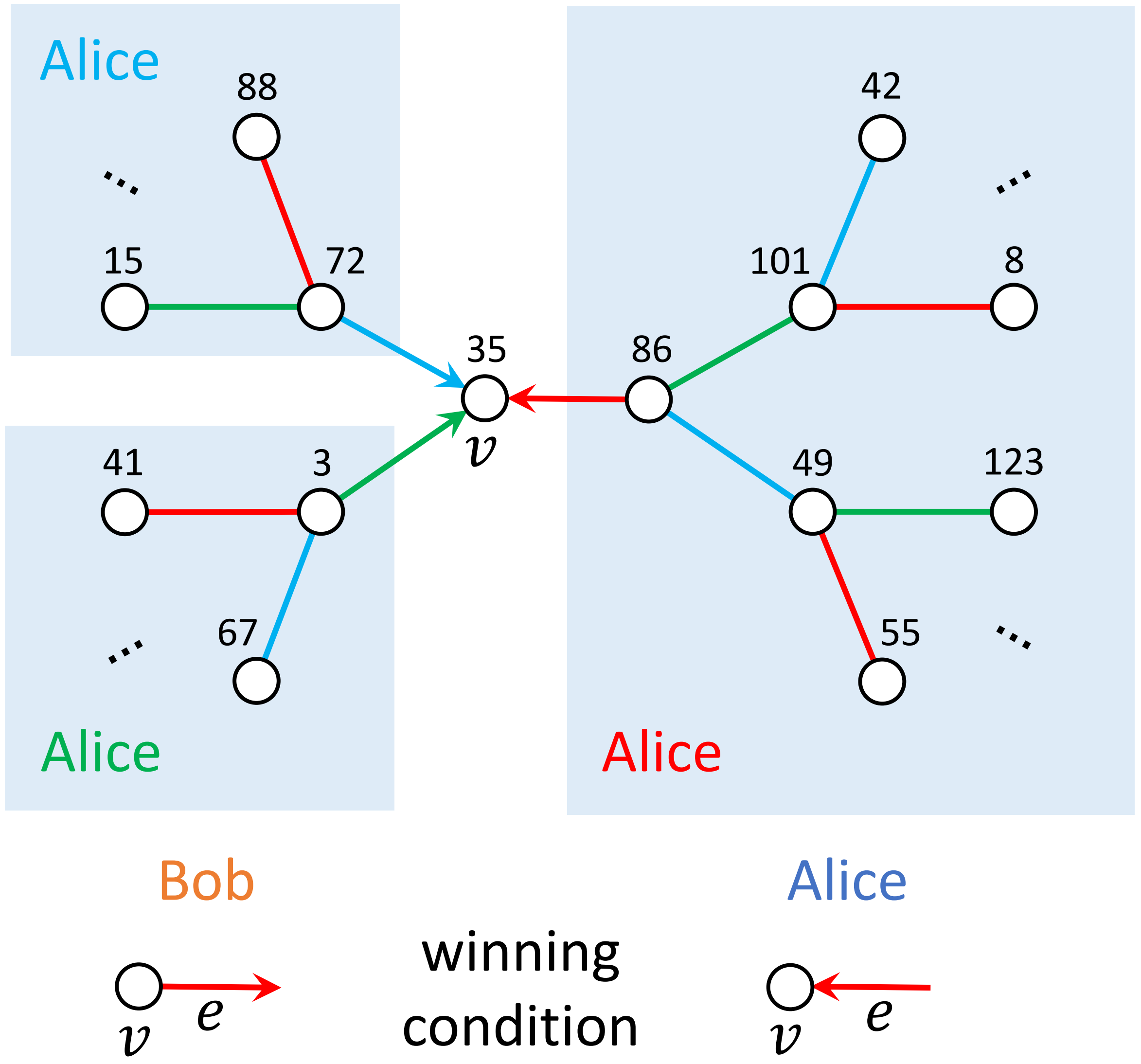
show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect

Can Alice win all of $(35, \text{red})$, $(35, \text{green})$, and $(35, \text{blue})$?

Let's play Alice's strategies against each other!

Marks' Technique

Sinkless Orientation



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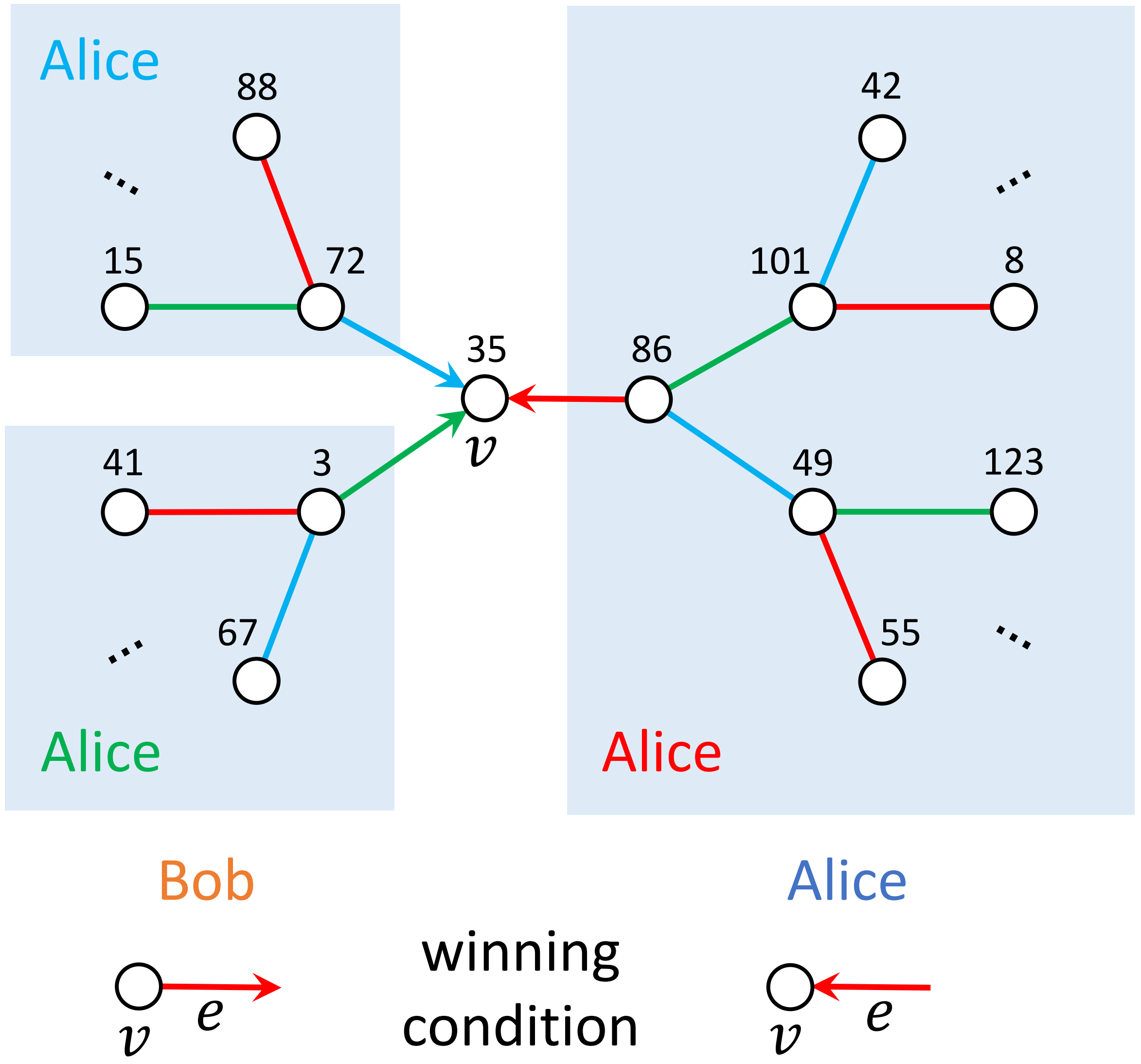

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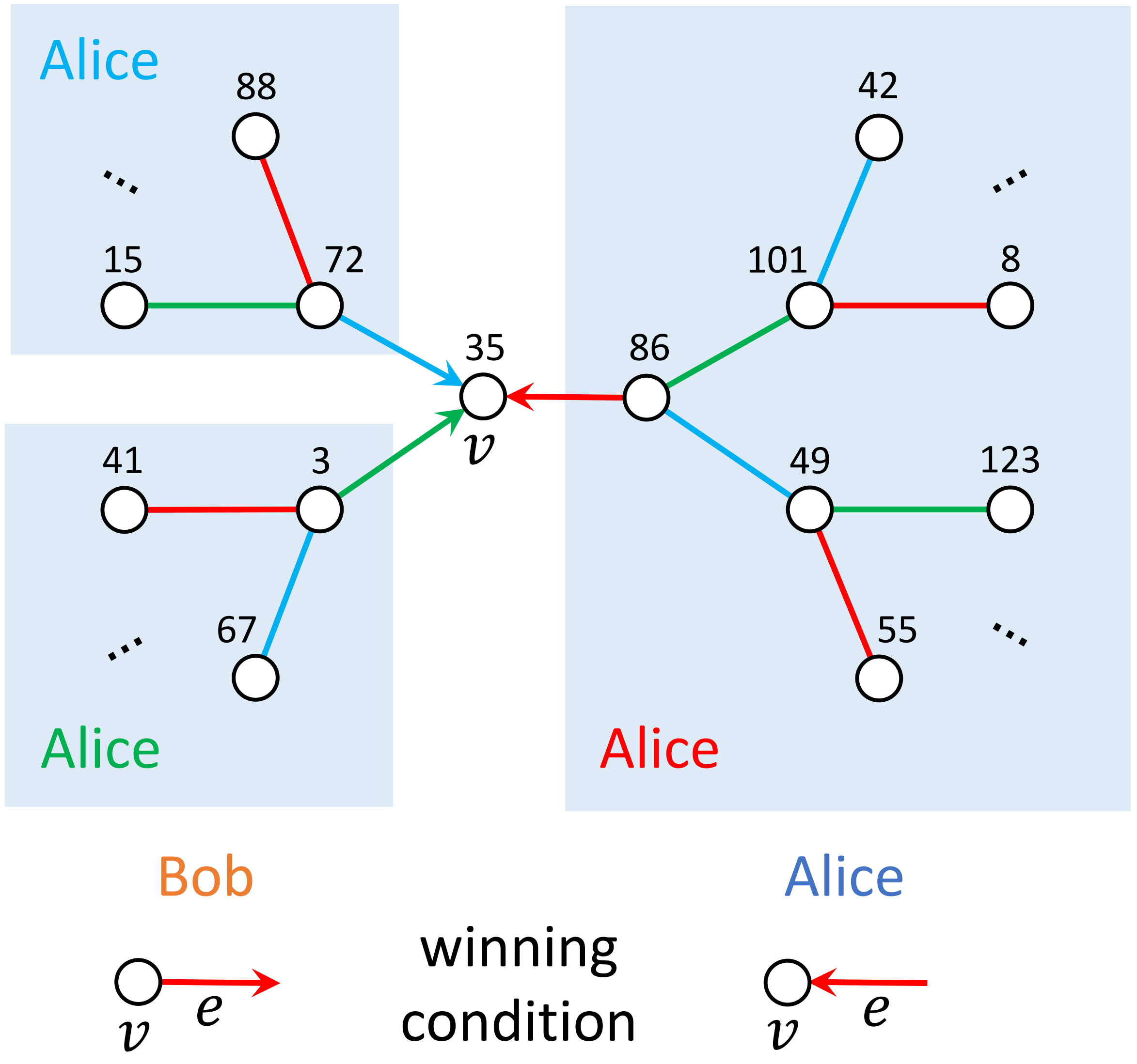

	red	green	blue
⋮			
35			
36			
37			
38			
⋮			

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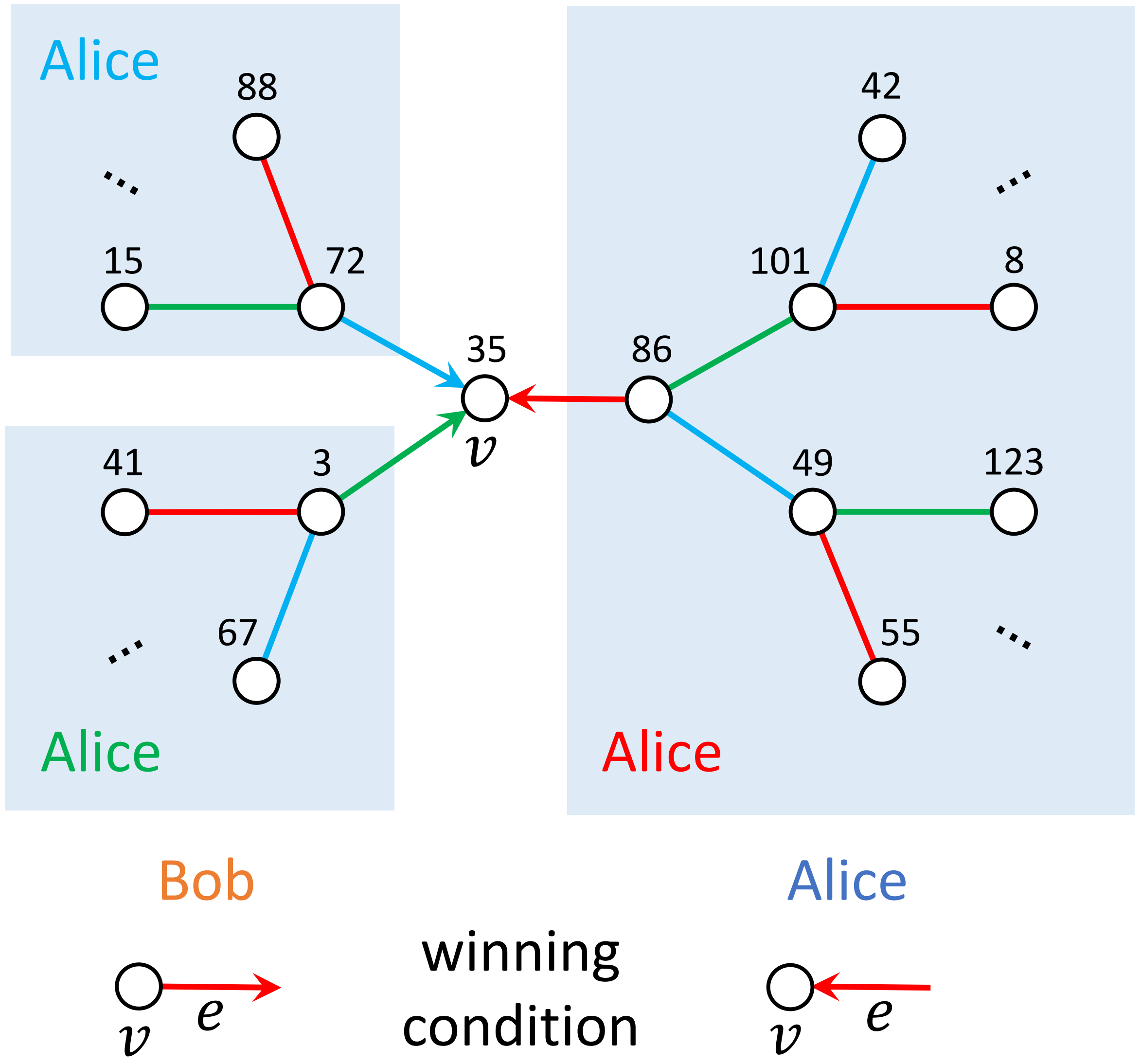

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⋮			
35	B	A	A
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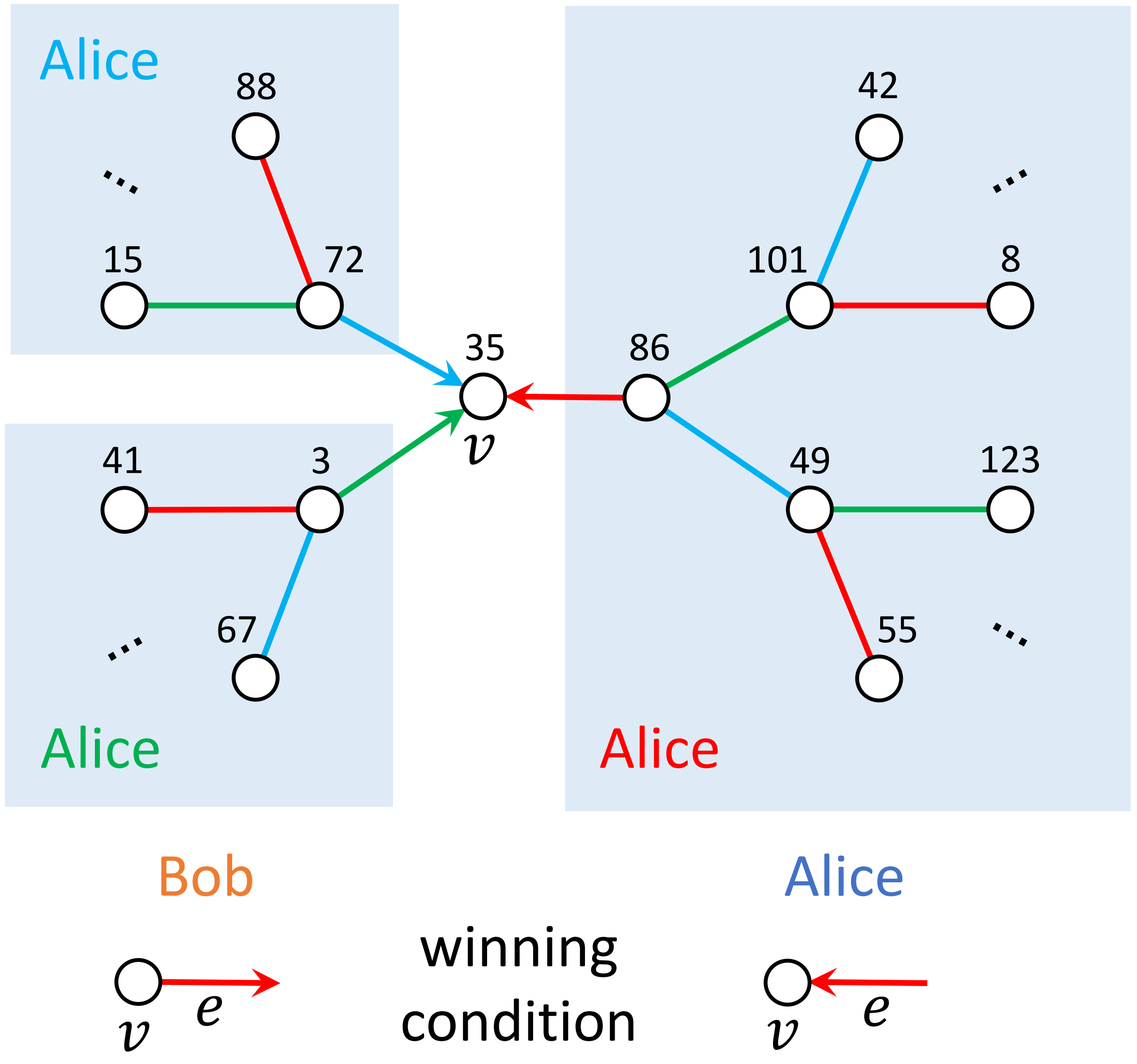

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Marks' Technique

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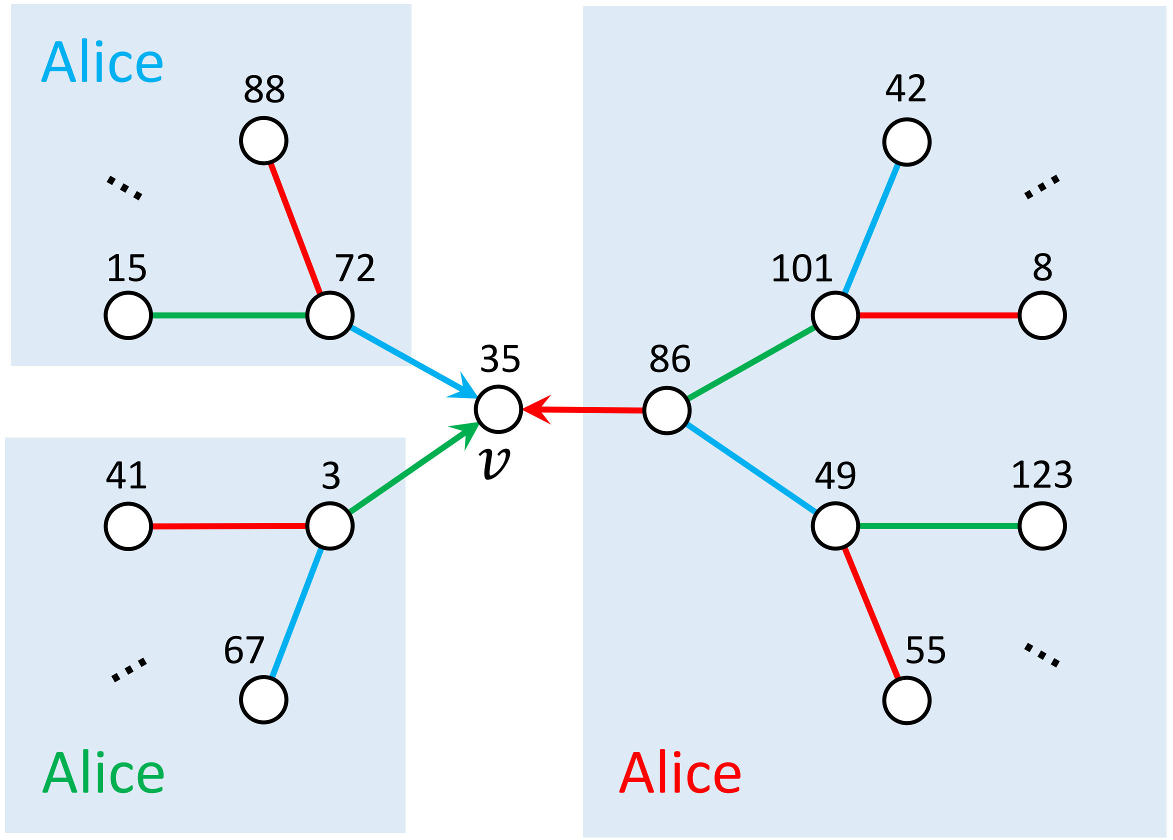

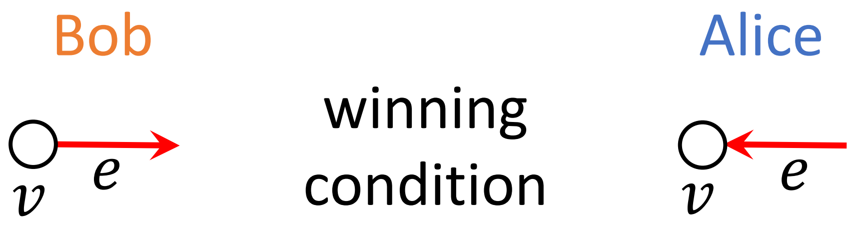

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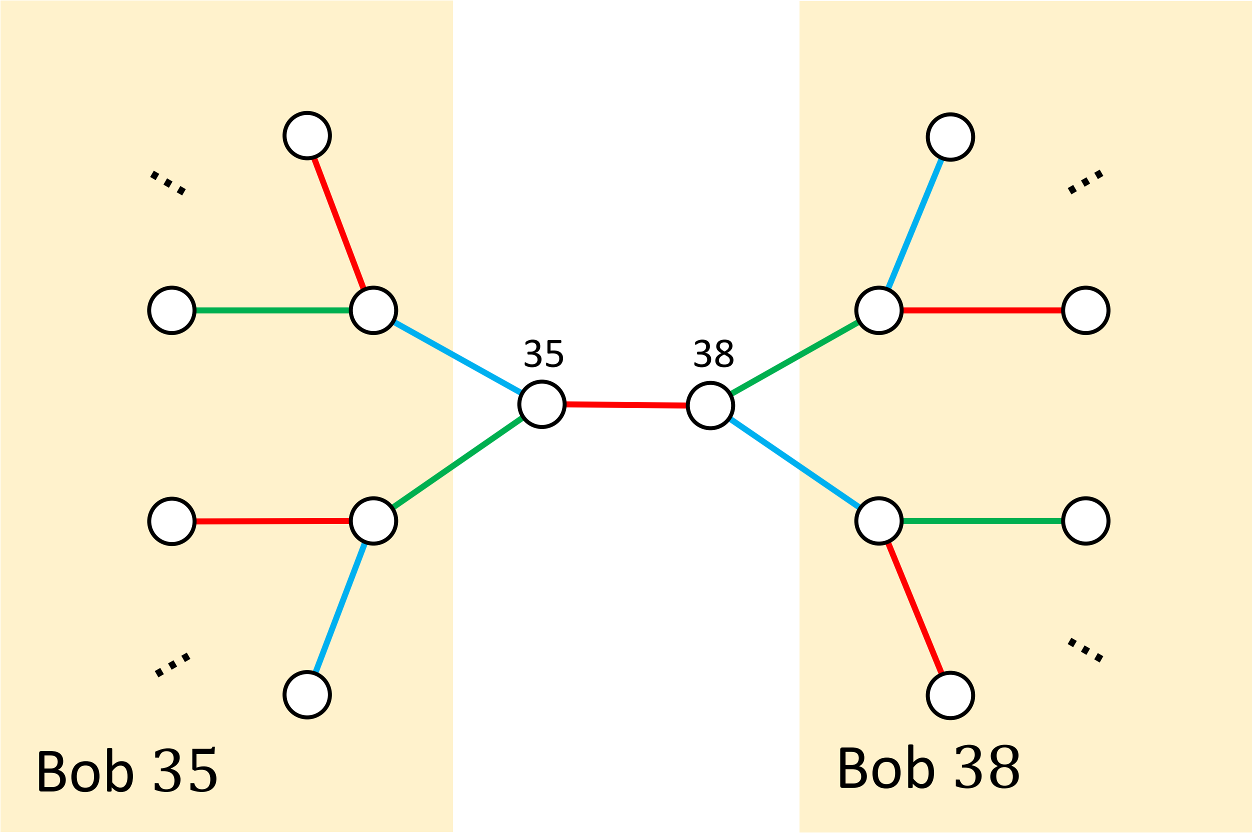
show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect

Can Bob win both of $(35, \mathrm{red})$ and $(38, \mathrm{red})$?

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation



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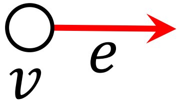
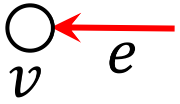
show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect

Bob



winning condition

Alice

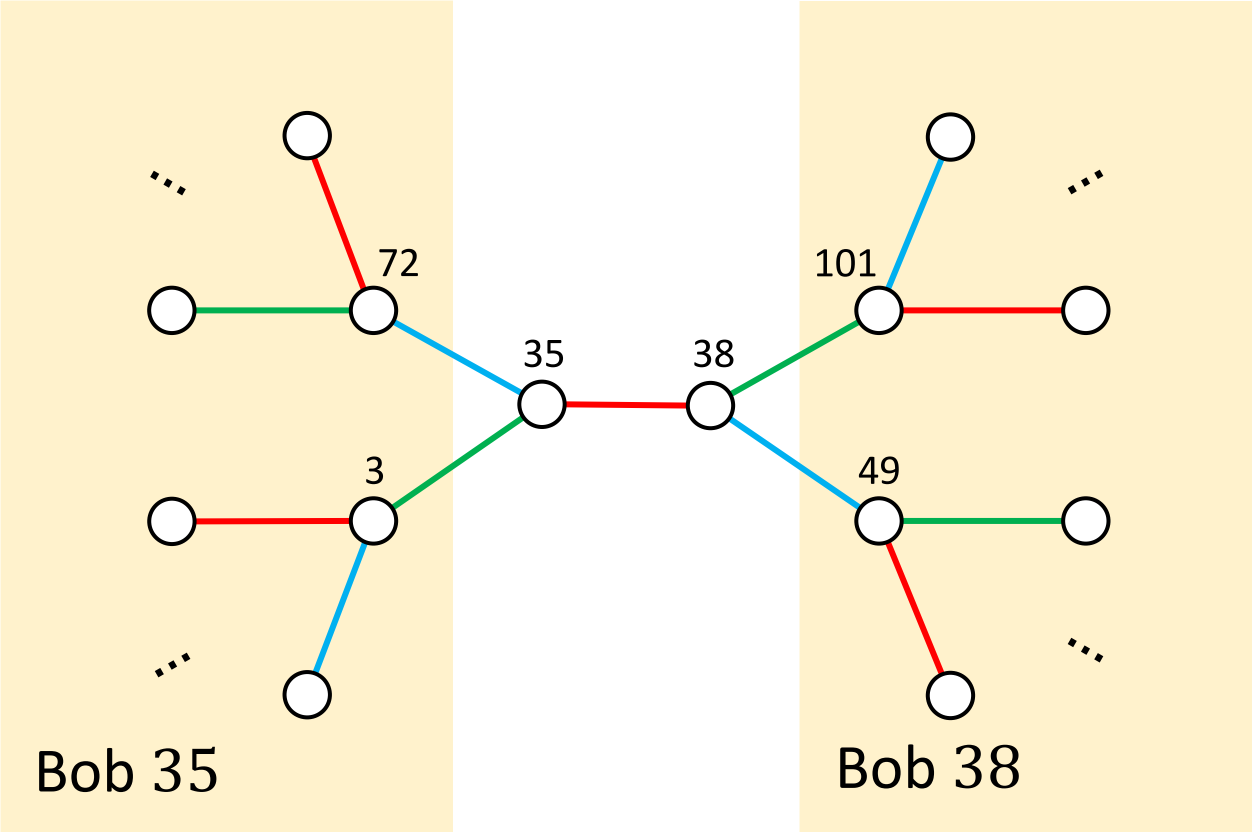


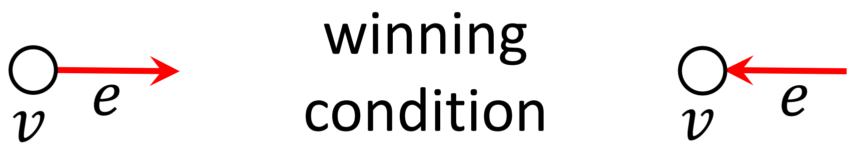
Can Bob win both of $(35, \mathrm{red})$ and $(38, \mathrm{red})$?

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation





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show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect

Can Bob win both of $(35, \mathrm{red})$ and $(38, \mathrm{red})$?

Let's play Bob's strategies against each other!

Marks' Technique

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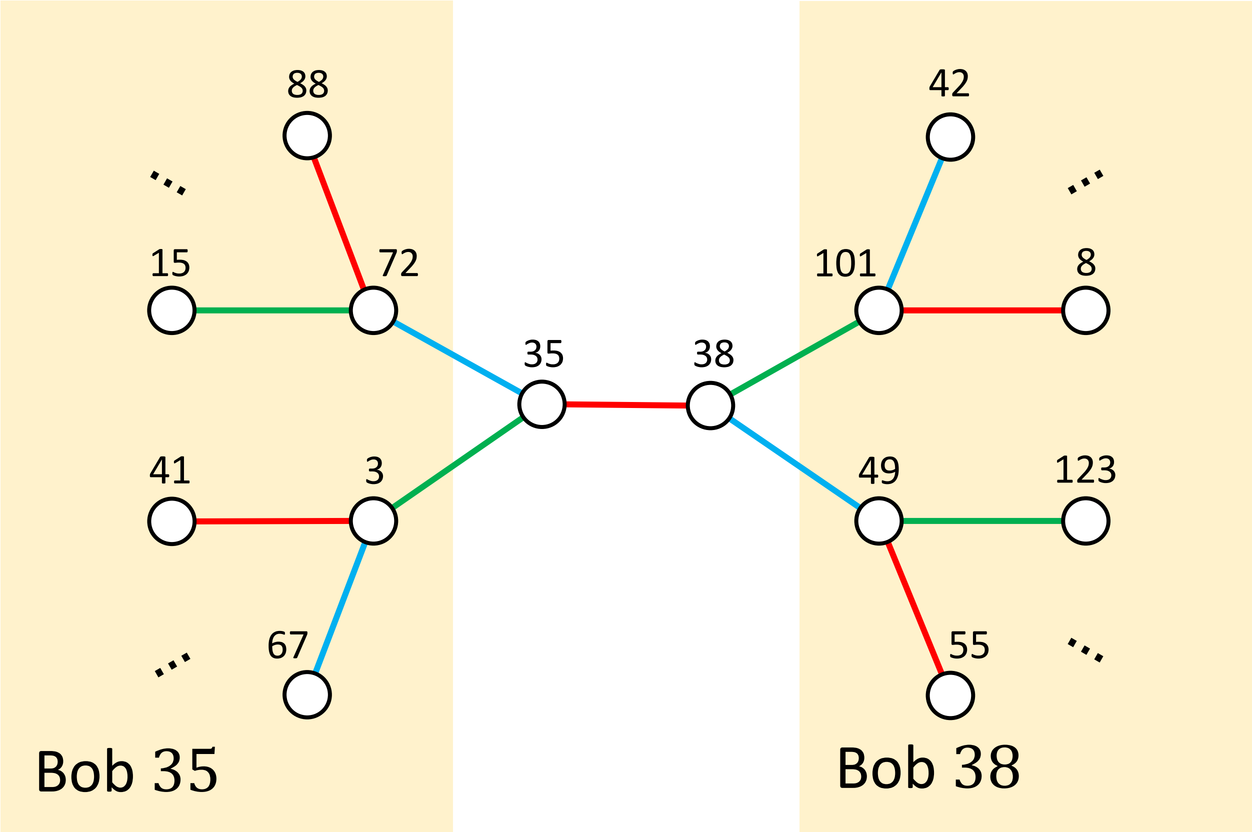

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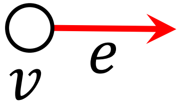
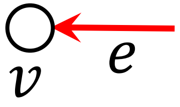


winning condition

Alice

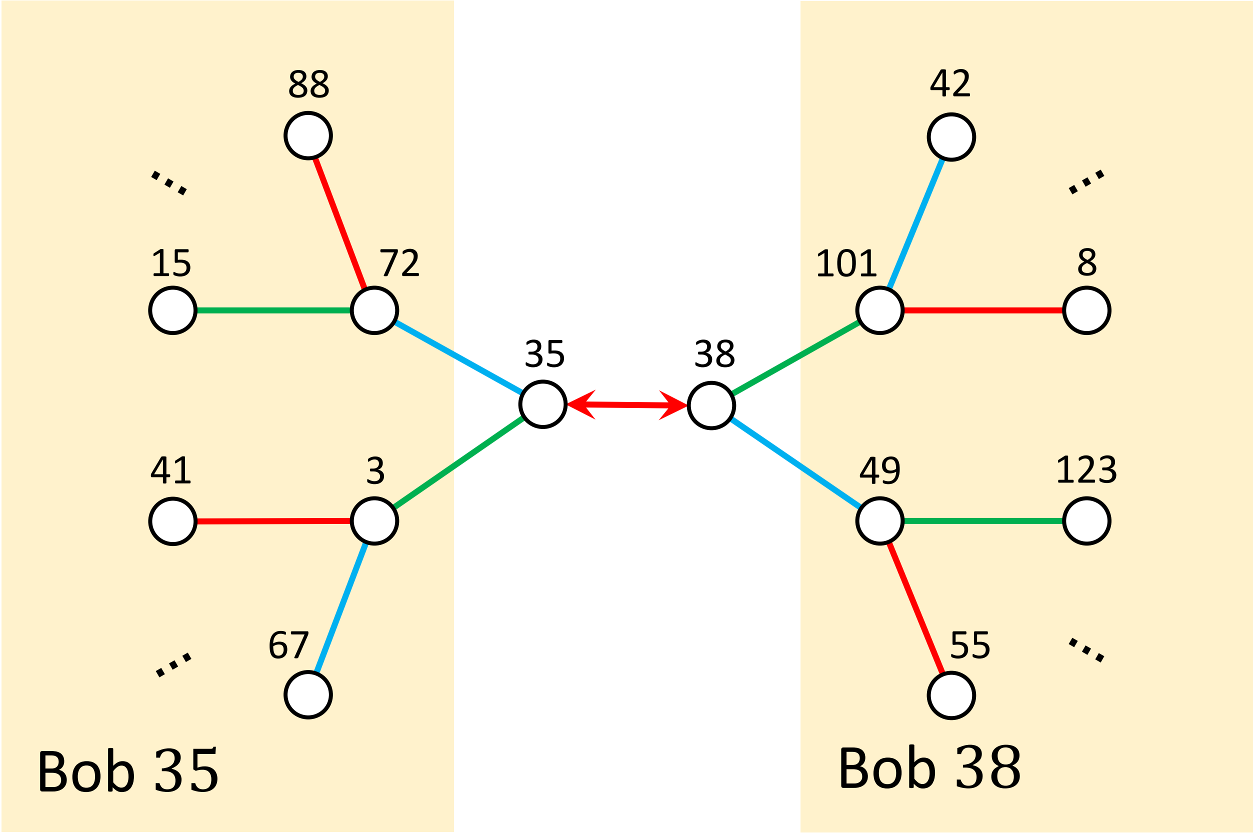


Can Bob win both of $(35, \text{red})$ and $(38, \text{red})$?

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation



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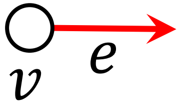
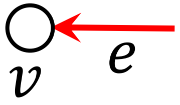
show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect

Bob



winning condition

Alice

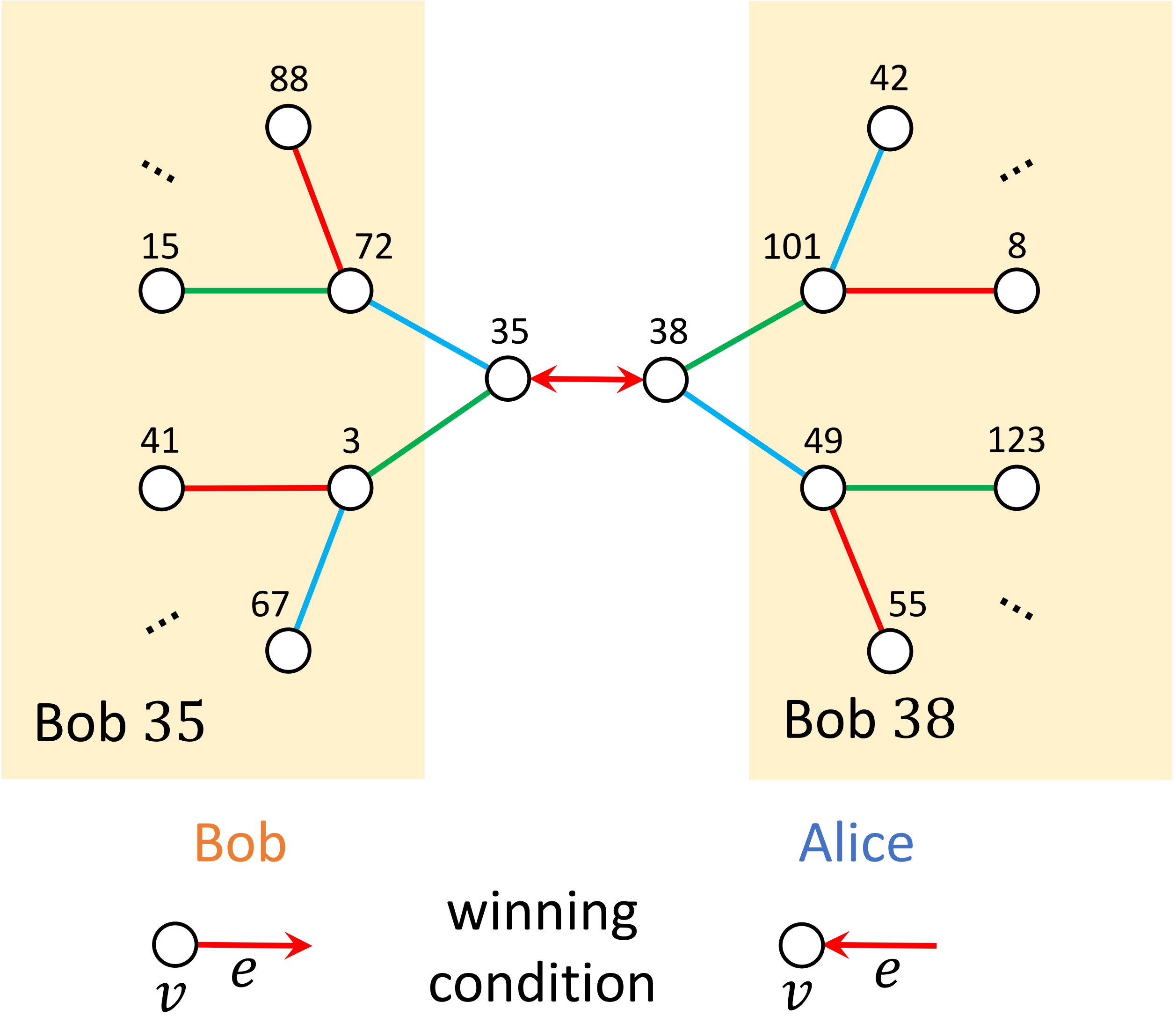


Can Bob win both of $(35, \mathrm{red})$ and $(38, \mathrm{red})$?

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation



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Can Bob win both of $(35, \text{red})$ and $(38, \text{red})$?

No!

Let's play Bob's strategies against each other!

Marks' Technique

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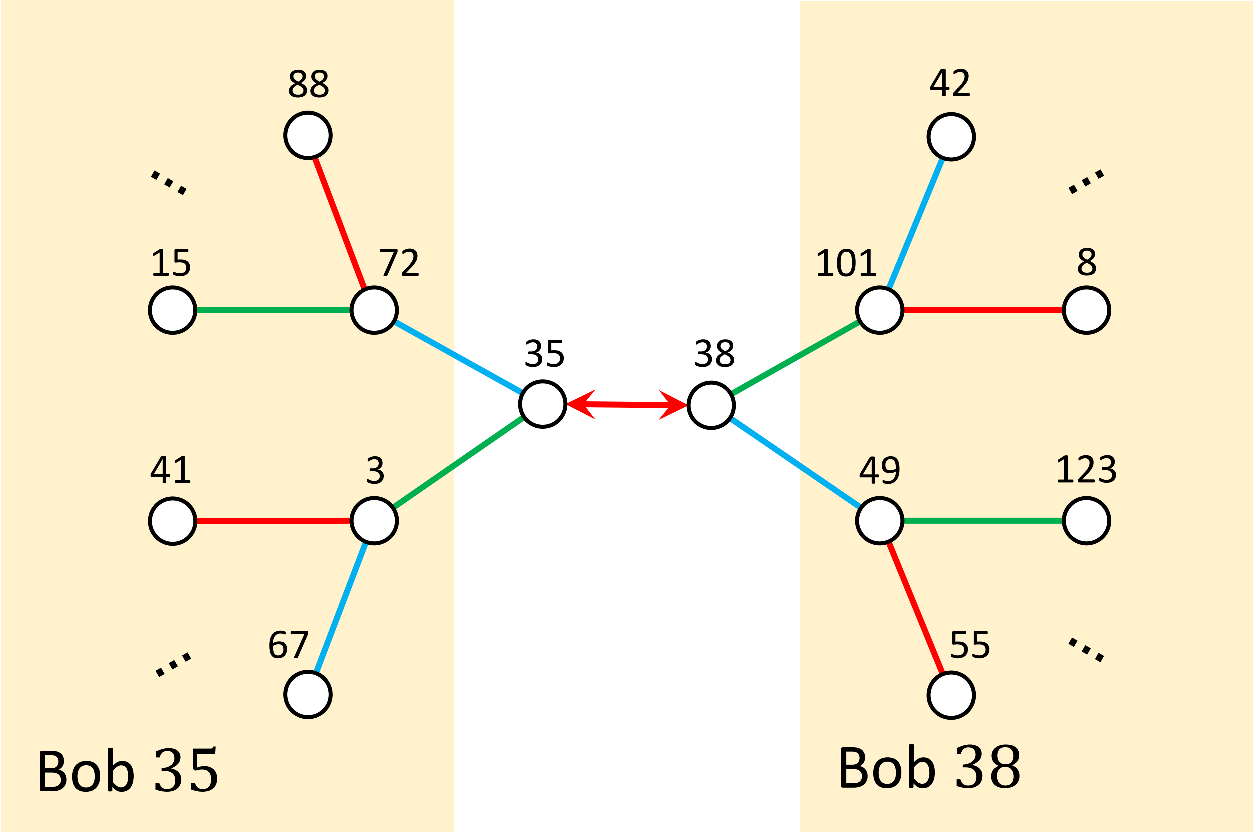
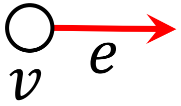


Bob



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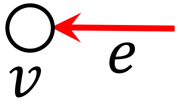

	red	green	blue
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show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect ✓

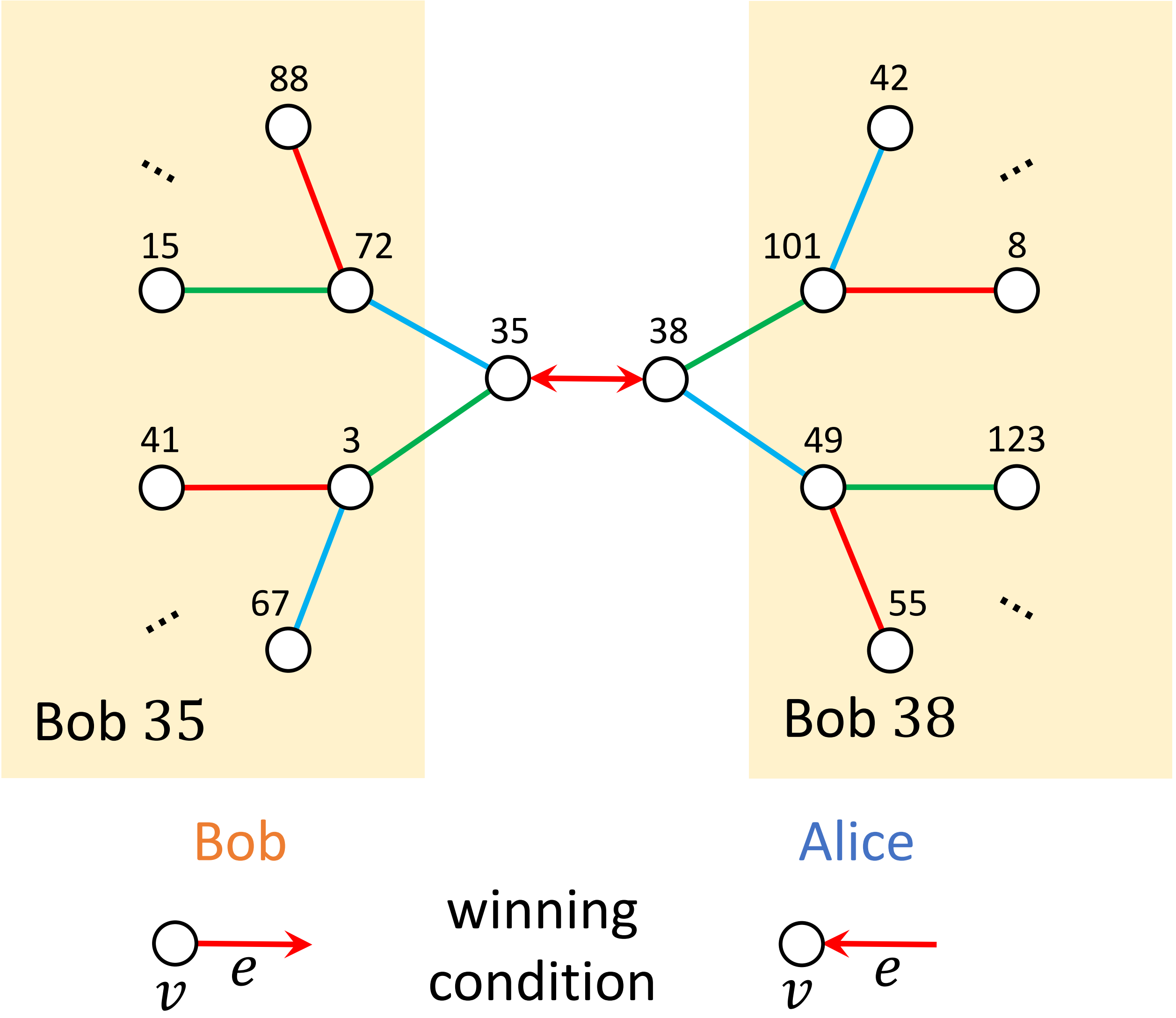
Can Bob win both of $(35, \text{red})$ and $(38, \text{red})$?

No!

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation



assume that there is a $o(\log n)$-round algorithm $\mathcal{A}$

↓

define a set of two-player games based on $\mathcal{A}$

↓

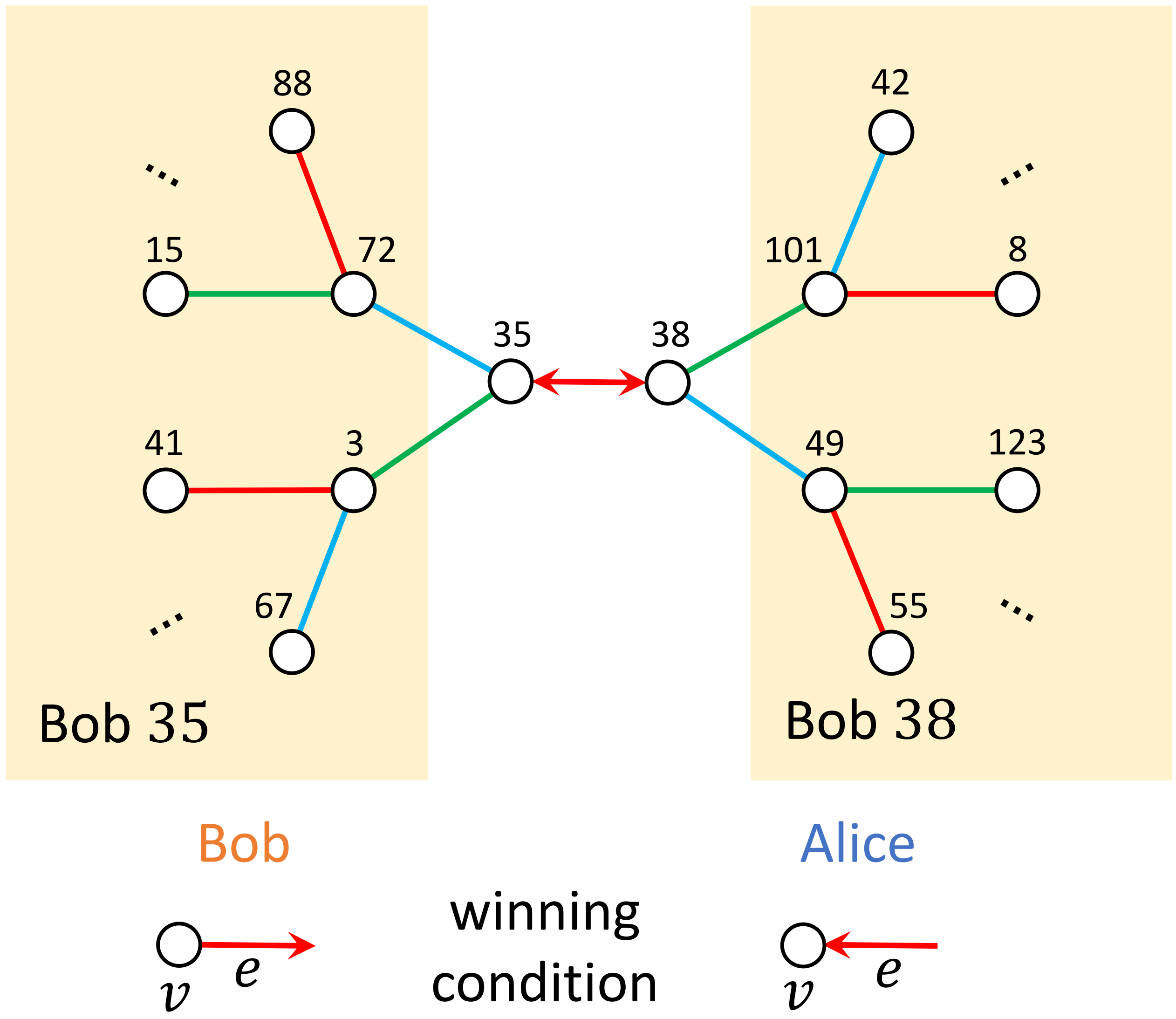
show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect ✓

Can Bob win both of $(35, \text{red})$ and $(38, \text{red})$? No!

Let's play Bob's strategies against each other!

Marks' Technique

Sinkless Orientation



assume that there is a $o(\log n)$-round algorithm $\mathcal{A}$ — false!

↓

define a set of two-player games based on $\mathcal{A}$

↓

show: each possible distribution of winning strategies implies that $\mathcal{A}$ is incorrect ✓

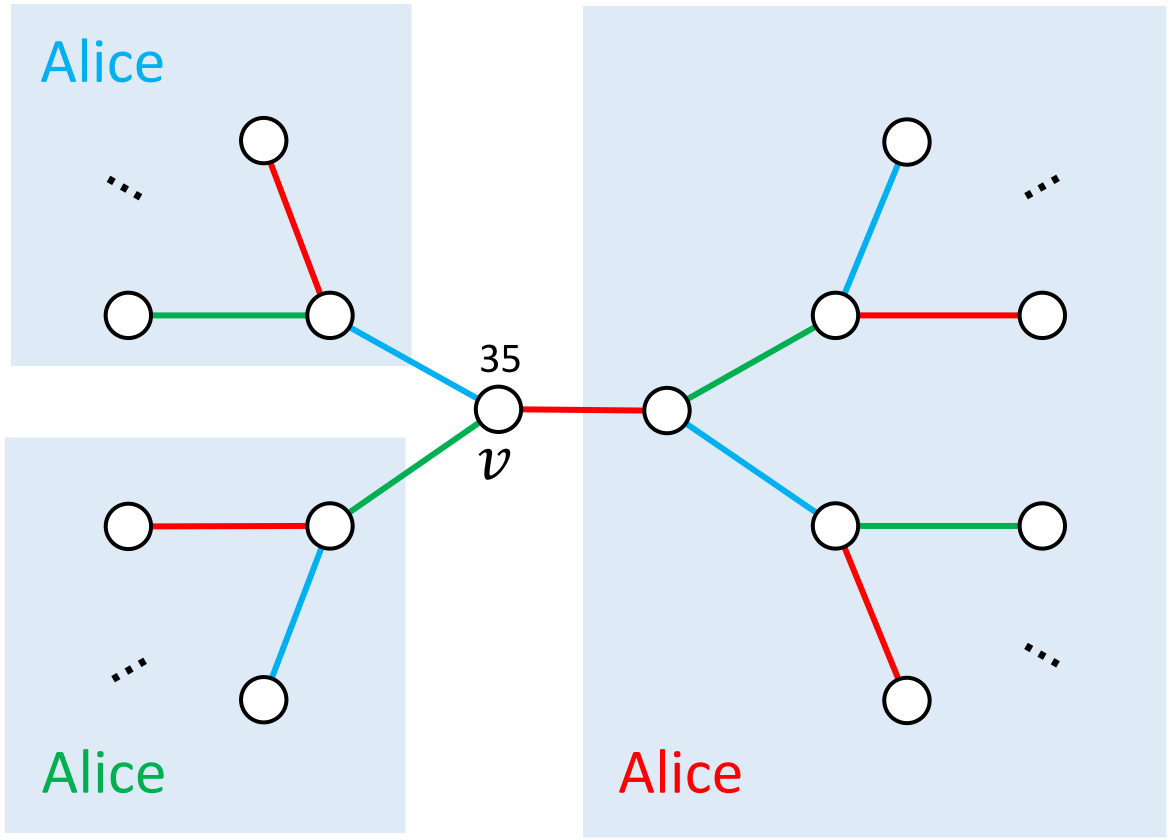
Can Bob win both of $(35, \text{red})$ and $(38, \text{red})$? — No!

Let's play Bob's strategies against each other!

Any algorithm solving Sinkless Orientation requires $\Omega(\log n)$ rounds.

A Technical Issue

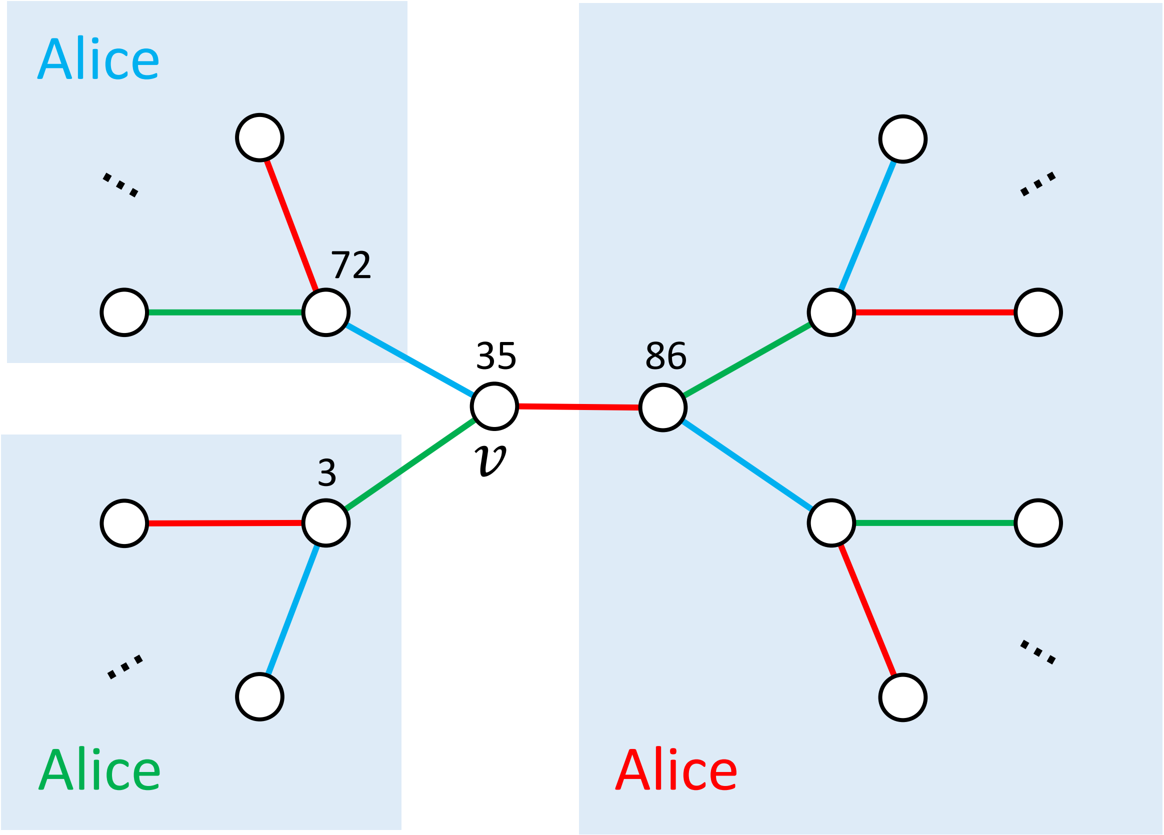
A Technical Issue



Can Alice win all of $(35, \text{red})$, $(35, \text{green})$, and $(35, \text{blue})$?

Let's play Alice's strategies against each other!

A Technical Issue



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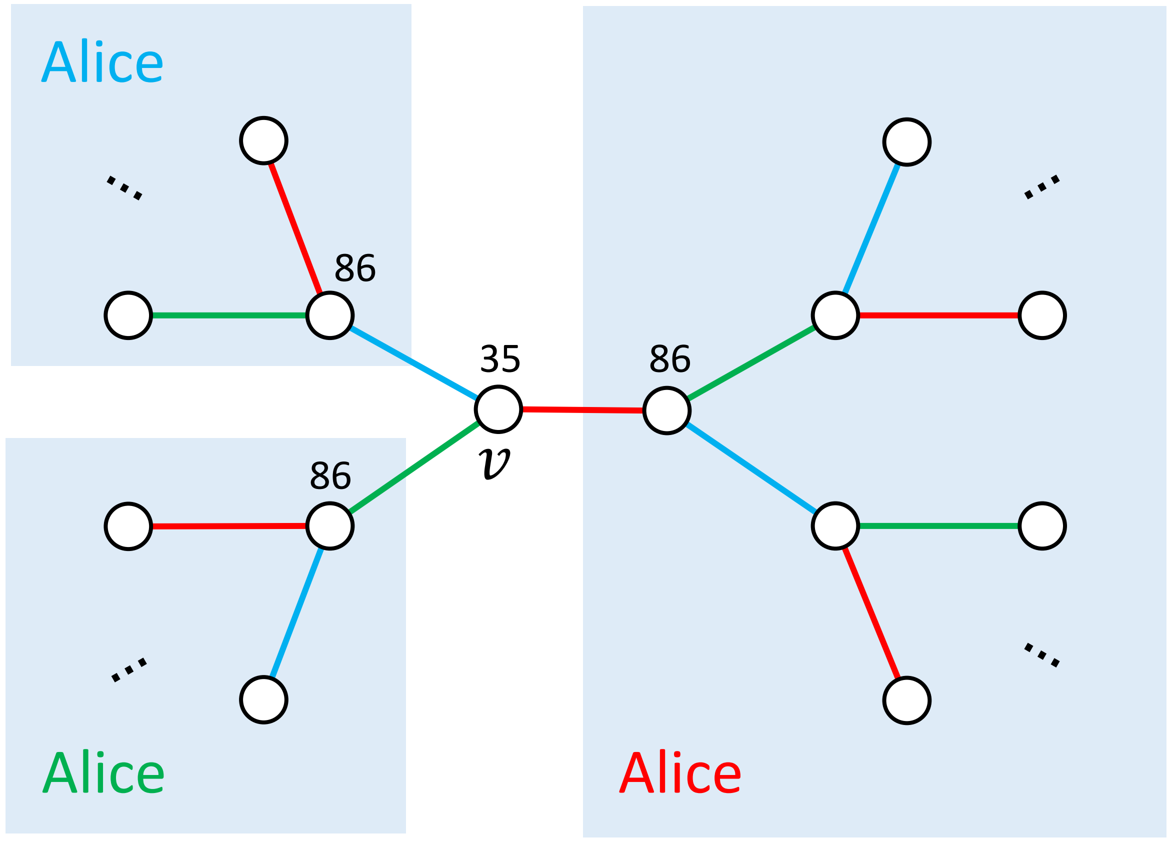

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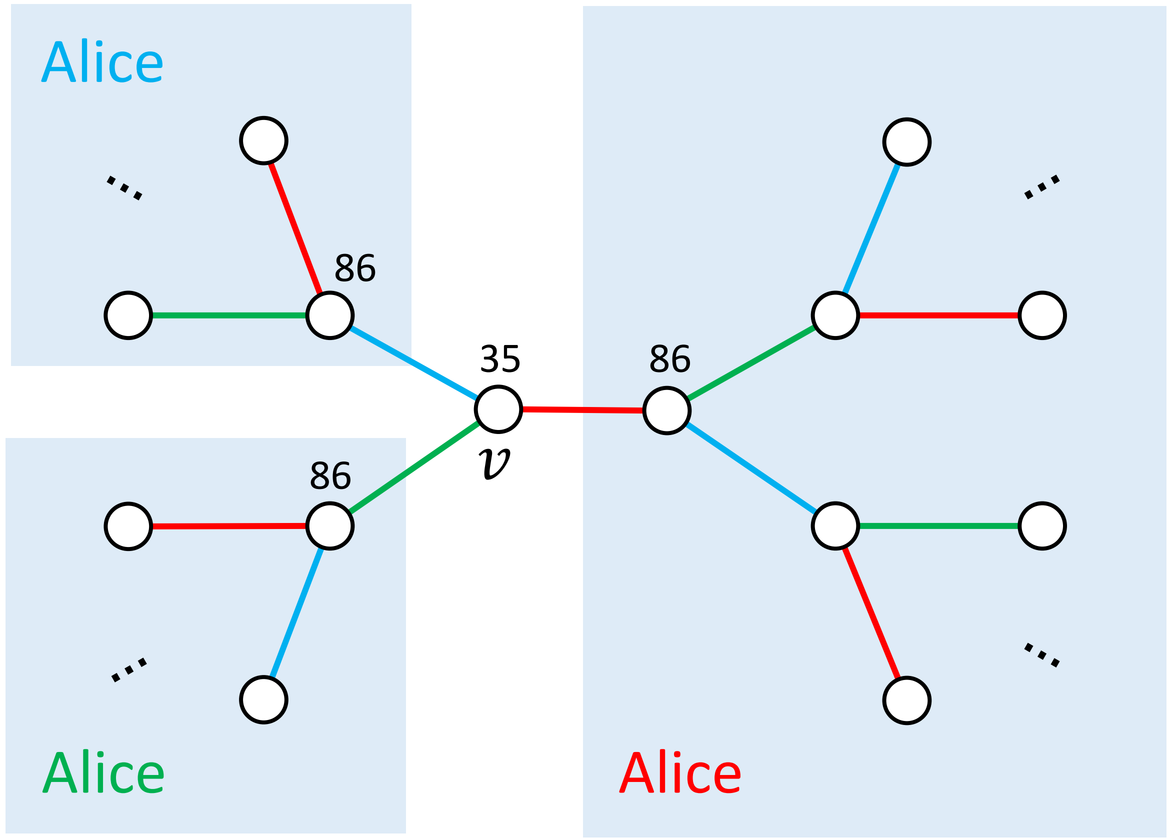
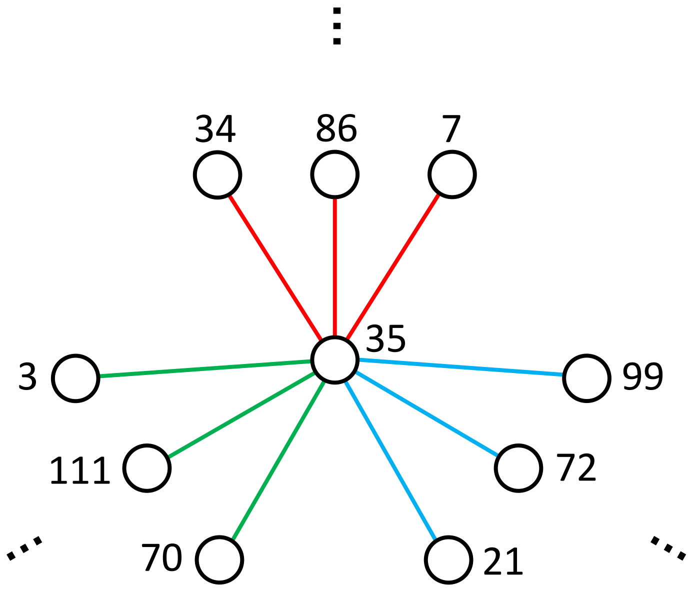


Solution: ID graph
(high-girth, node set = ID set)



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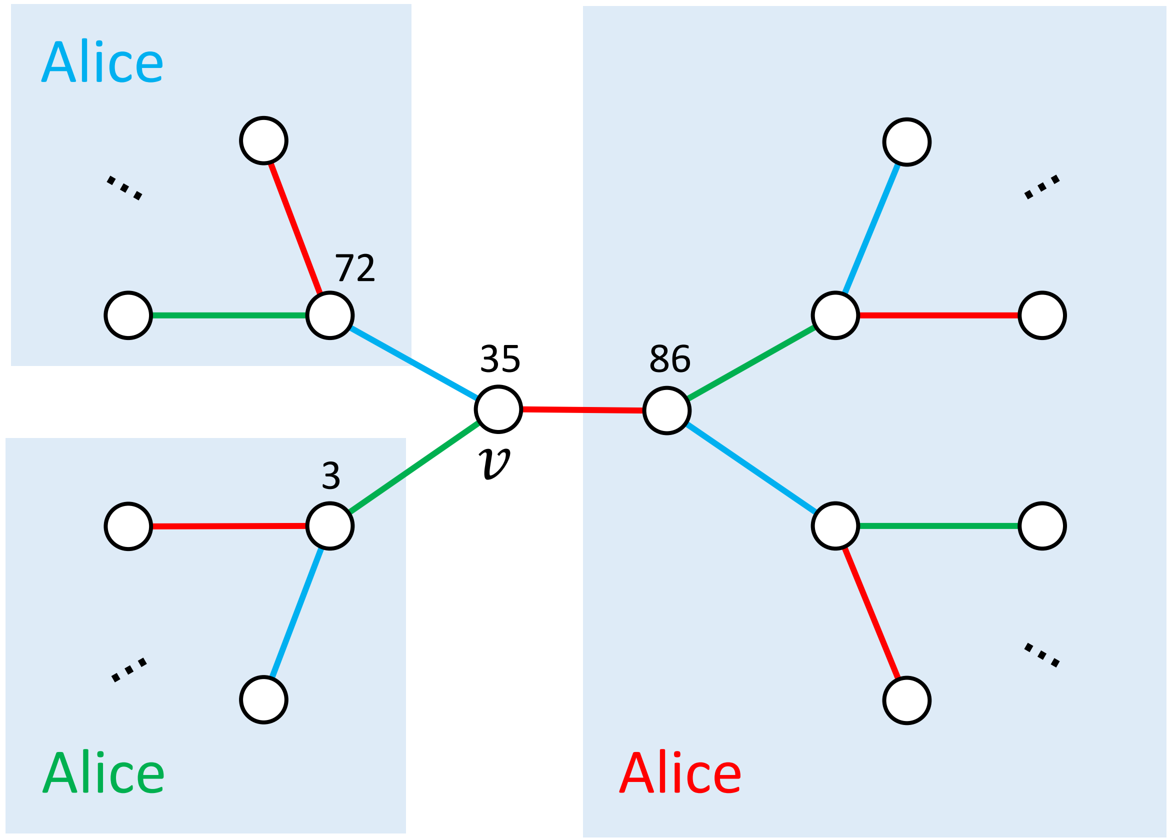

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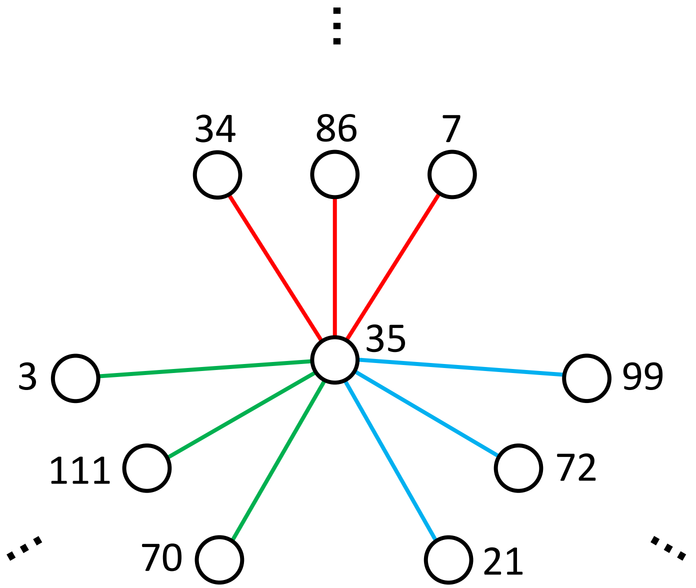

Can Alice win all of $(35, \text{red})$, $(35, \text{green})$, and $(35, \text{blue})$?

Let's play Alice's strategies against each other!

A Technical Issue

	red	green	blue
⋮			
35	B	A	A
36	A	A	B
37	A	B	A
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⋮			

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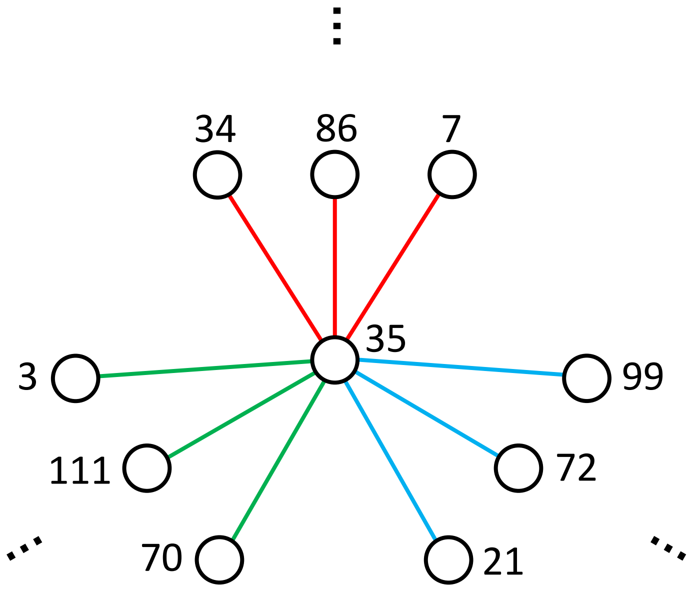
(high-girth, node set = ID set)



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A Technical Issue

	red	green	blue
⋮			
35	B	A	A
36	A	A	B
37	A	B	A
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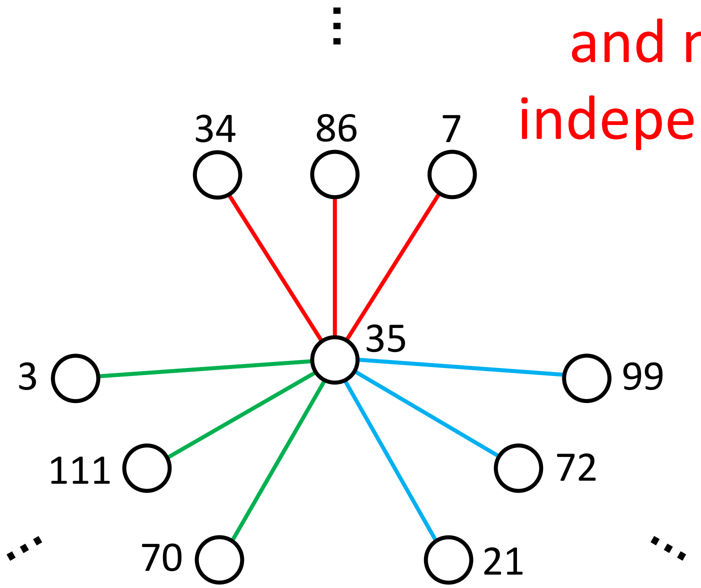
⋮			

Solution: ID graph
(high-girth, node set = ID set)

and no large independent set



Can Alice win all of $(35, \text{red})$, $(35, \text{green})$, and $(35, \text{blue})$?

Let's play Alice's strategies against each other!

A Technical Issue

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \quad \dots \quad \xrightarrow{\mathcal{R}} \Pi_k$$

T_0 $\quad$ $T_1 = T_0 - 1$ $\quad$ $T_2 = T_1 - 1$ $\quad$ $T_k > 0$

Solution: ID graph

A Technical Issue

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \quad \dots \quad \xrightarrow{\mathcal{R}} \Pi_k$$

$$T_0 \qquad T_1 = T_0 - 1 \qquad T_2 = T_1 - 1 \qquad \qquad T_k > 0$$

Solution: ID graph

Solution 2: randomness

A Technical Issue

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \cdots \xrightarrow{\mathcal{R}} \Pi_k$$

T_0 $\quad$ $T_1 = T_0 - 1$ $\quad$ $T_2 = T_1 - 1$ $\quad$ $T_k > 0$

p_0 $\quad$ $p_1 \geq p_0$ $\quad$ $p_2 \geq p_1$ $\quad$ p_k

prove a lower bound in the randomized LOCAL model

A Technical Issue

$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \xrightarrow{\mathcal{R}} \cdots \xrightarrow{\mathcal{R}} \Pi_k$$

T_0	$T_1 = T_0 - 1$	$T_2 = T_1 - 1$	$T_k > 0$
p_0	$p_1 \geq p_0$	$p_2 \geq p_1$	p_k

prove a lower bound in the randomized LOCAL model

↓

lift the lower bound to the deterministic LOCAL model

A Technical Issue

Solution: ID graph

Solution 2: randomness

Different advantages
and disadvantages!

A Technical Issue

Is there a meaningful randomized version of Marks' technique?

Solution: ID graph

Solution 2: randomness

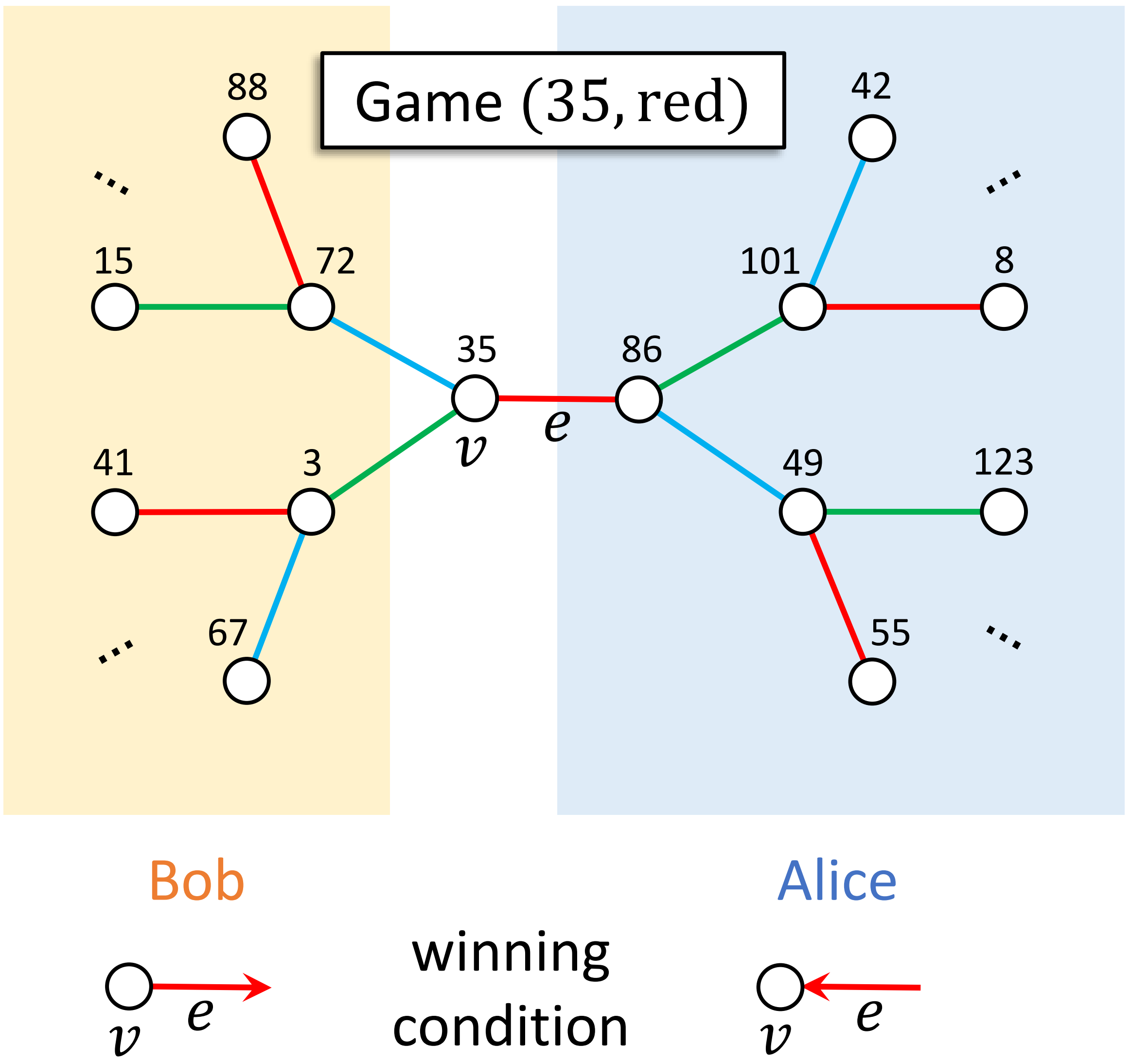
Different advantages and disadvantages!

Generalization of Marks' Technique

[Brandt, Chang, Grebík, Grunau, Rozhoň, Vidnyánszky, 2022]

Marks' Technique

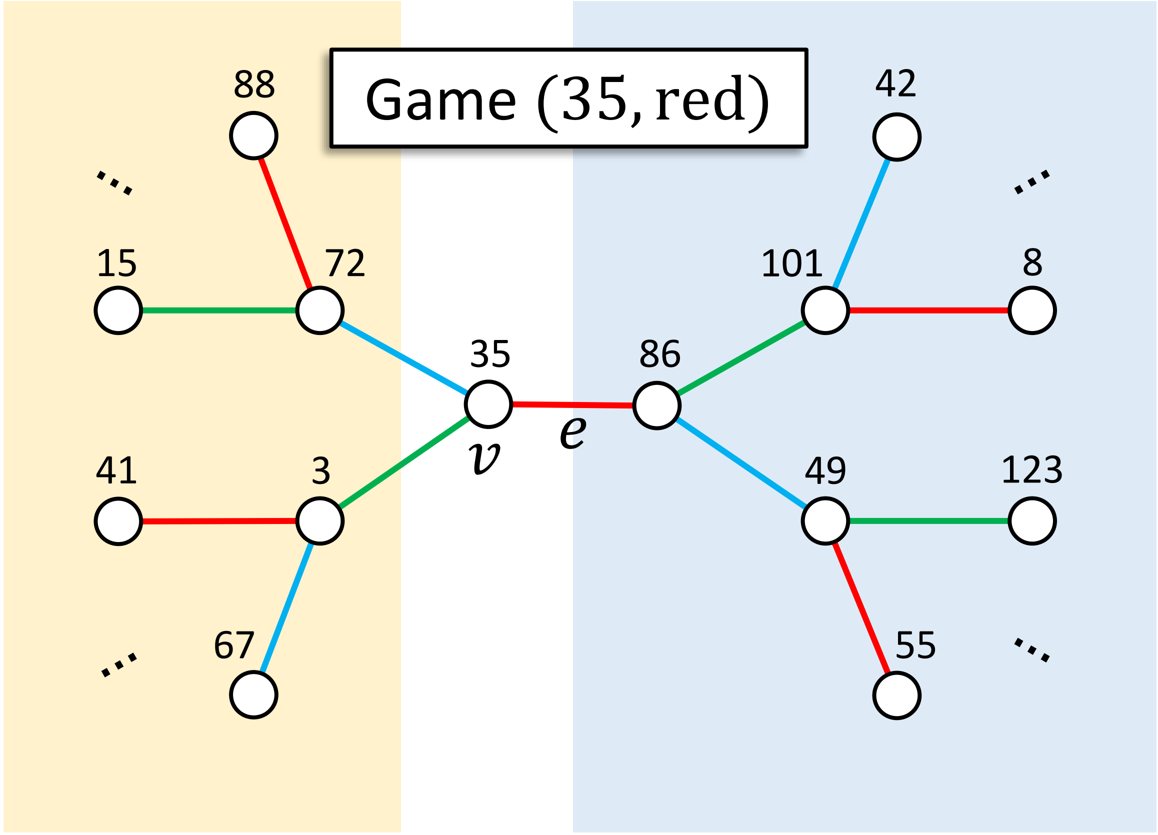
Sinkless Orientation



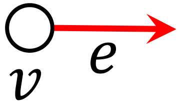
One game for each pair (ID, color).

Marks' Technique

any LCL problem

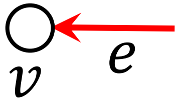


Bob



winning condition

Alice

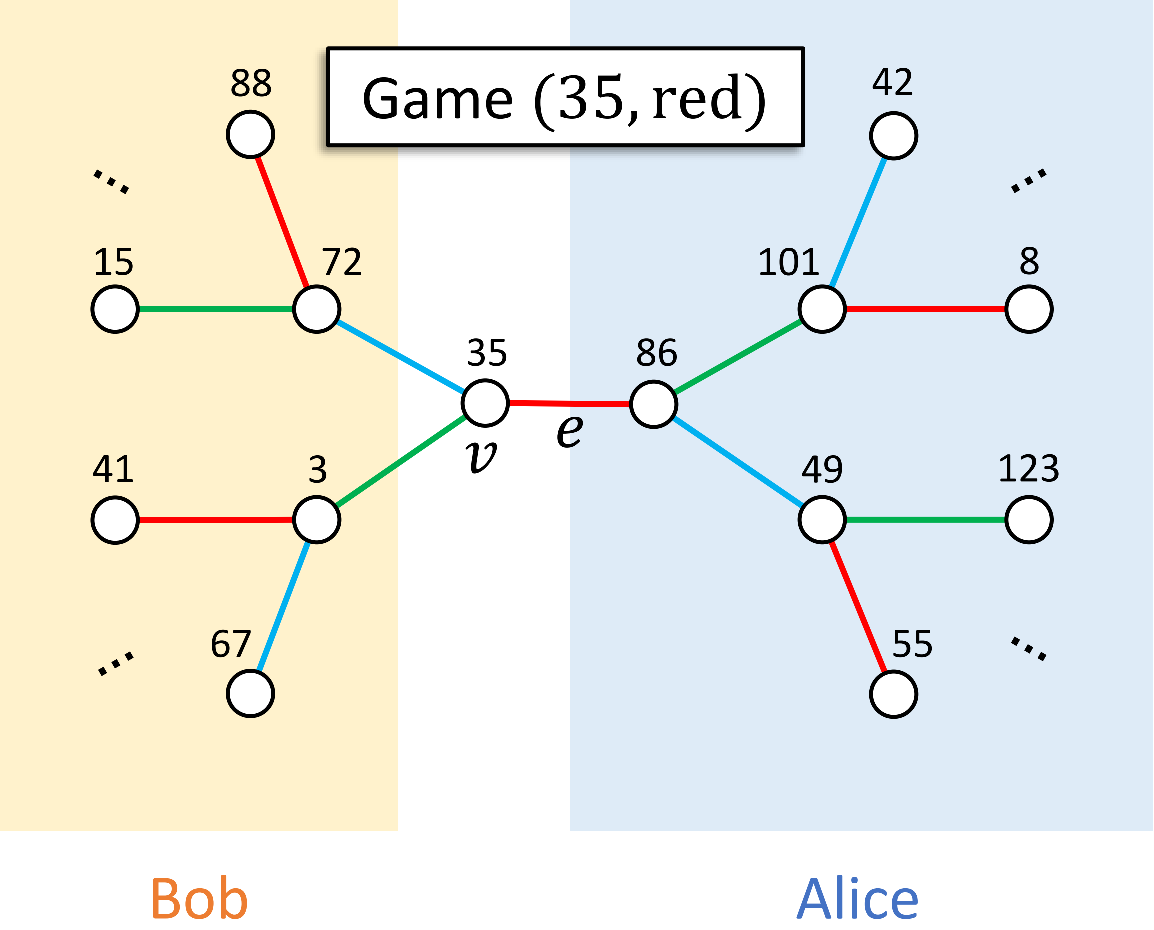


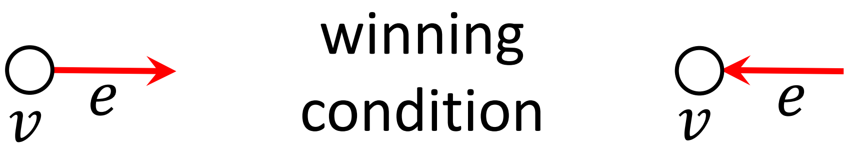
One game for each pair (ID, color).

Marks' Technique

any LCL problem

more games/winning conditions

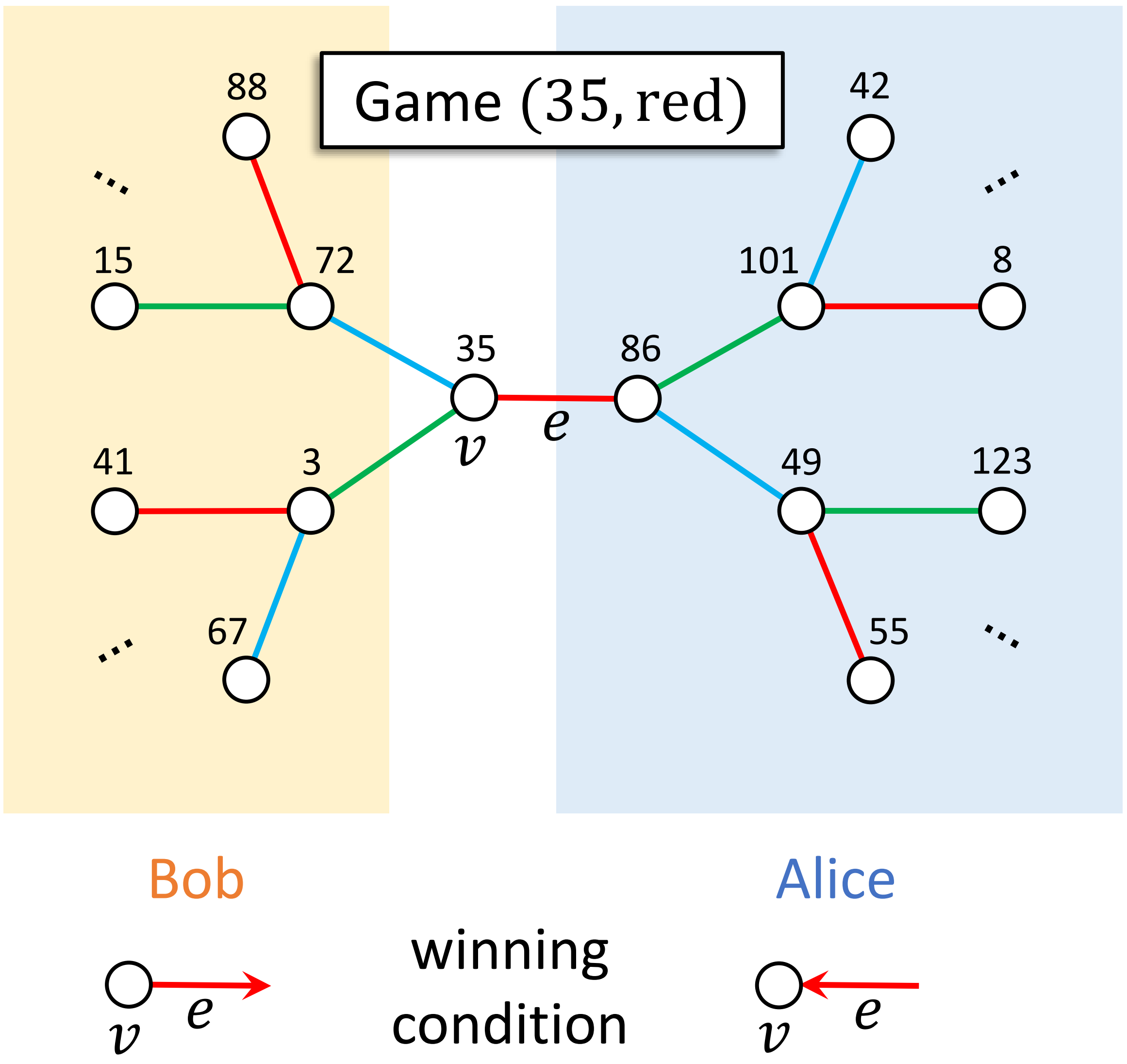




One game for each pair (ID, color).

Marks' Technique

any LCL problem



more games/winning conditions

Whether (this generalization of) Marks' Technique works can be characterized by a simple criterion.

One game for each pair (ID, color).

2 Labels.

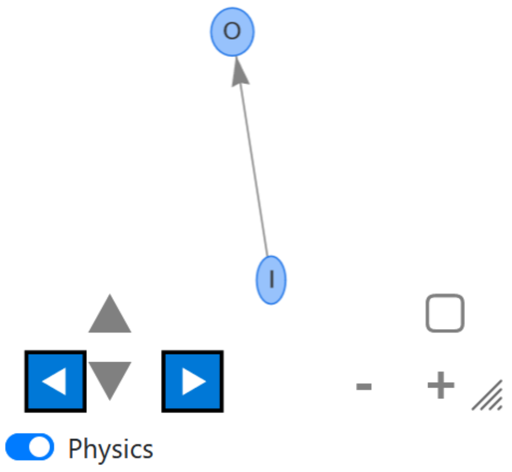
Active
Any choice satisfies previous Passive

I O

Passive
Exists choice satisfying previous Active

O IO 2

Partial Diagram
Strength of passive labels



Tools
Speedup, edit, simplifications, ...

Operations
(speedup, maximize, edit, gen renaming, merge)

Speedup apply round elimination

Maximize maximize passive side (and compute full diagram, triviality, ...)

Edit copy problem up

Inverse Speedup apply inverse round elimination

Speedup+Maximize Speedup+Maximize+Rename

1 Fix Outdegree

Coloring compute hypergraph strong coloring solvability

Apply Marks apply Marks' technique

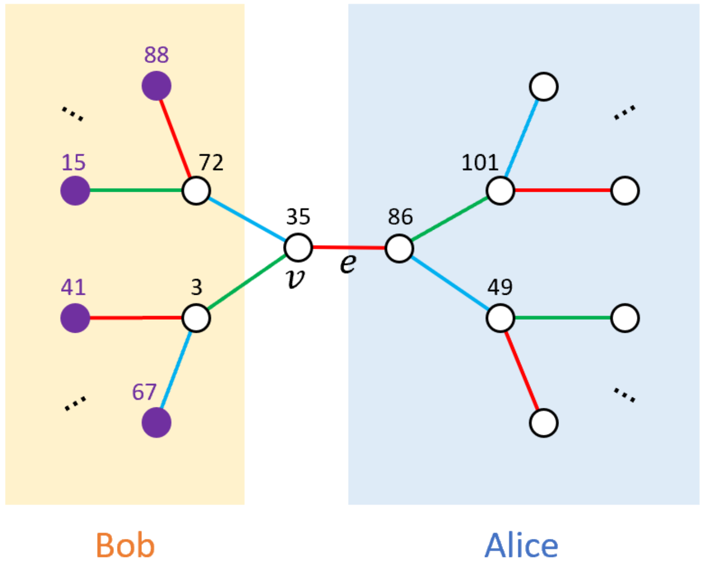
https://roundeliminator.github.io/re-experimental

Similarities

Round Elimination

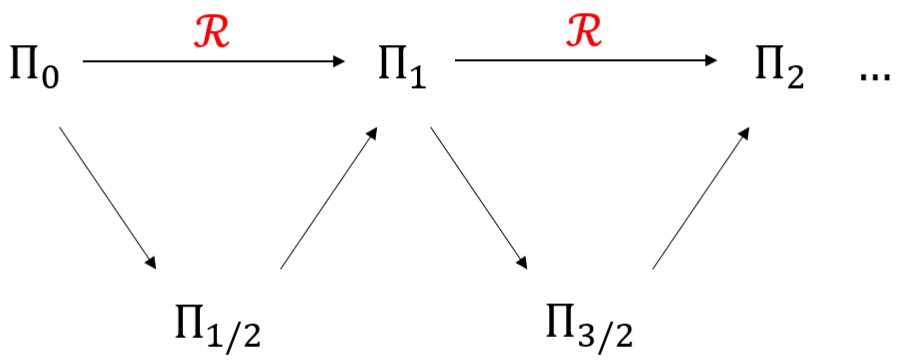
$$\Pi_0 \xrightarrow{\mathcal{R}} \Pi_1 \xrightarrow{\mathcal{R}} \Pi_2 \quad \ldots$$

Marks' Technique

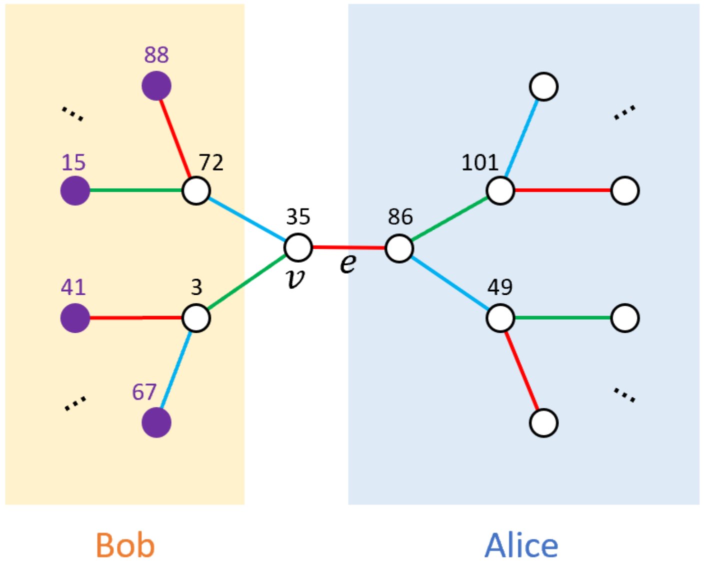


Similarities

Round Elimination



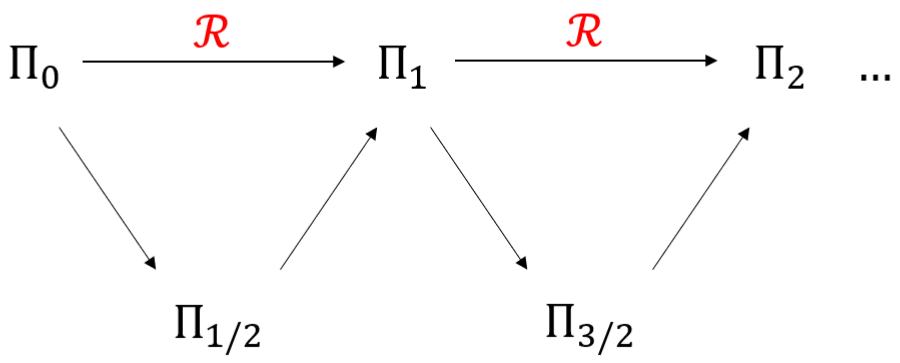
Marks' Technique



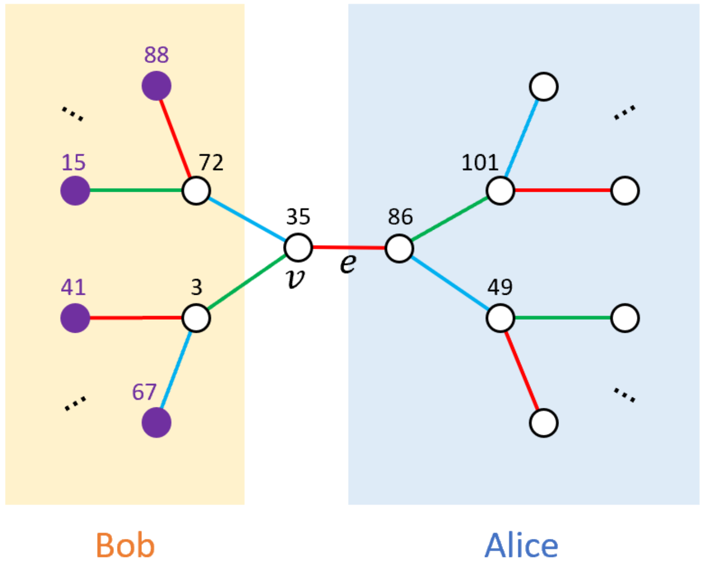
Similarities

- two players/kinds of algorithms,
  node- and edge-based considerations

Round Elimination



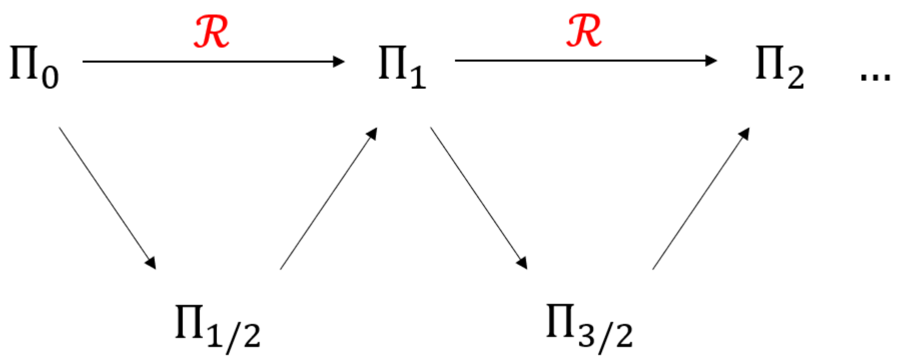
Marks' Technique



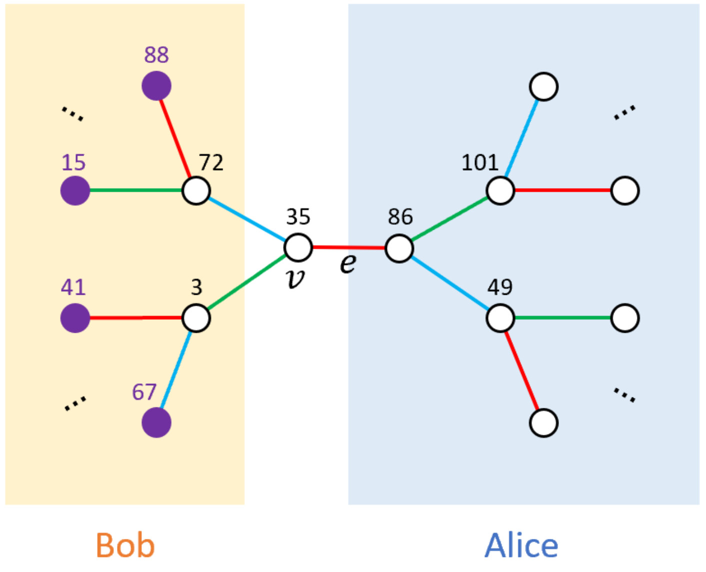
Similarities

- two players/kinds of algorithms, node- and edge-based considerations
- stepwise increase/decrease by "extensions" of neighborhoods

Round Elimination



Marks' Technique



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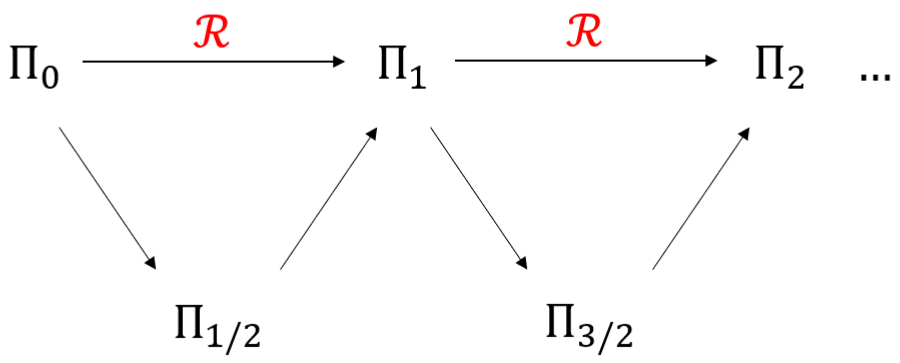
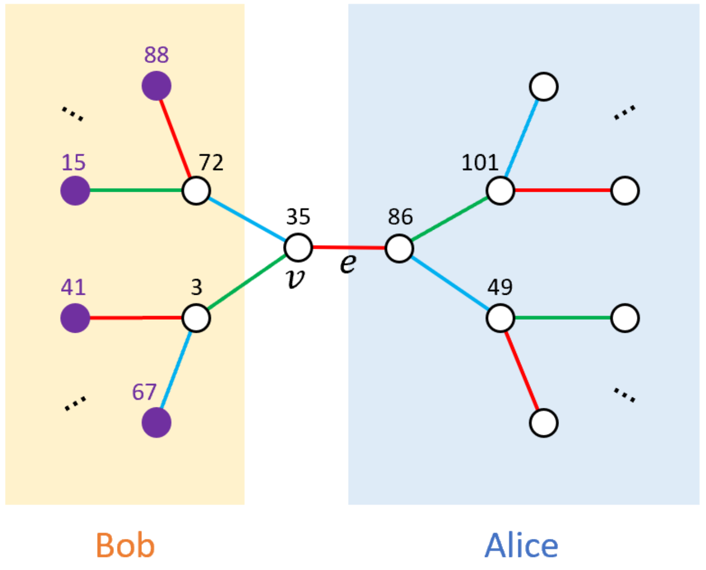
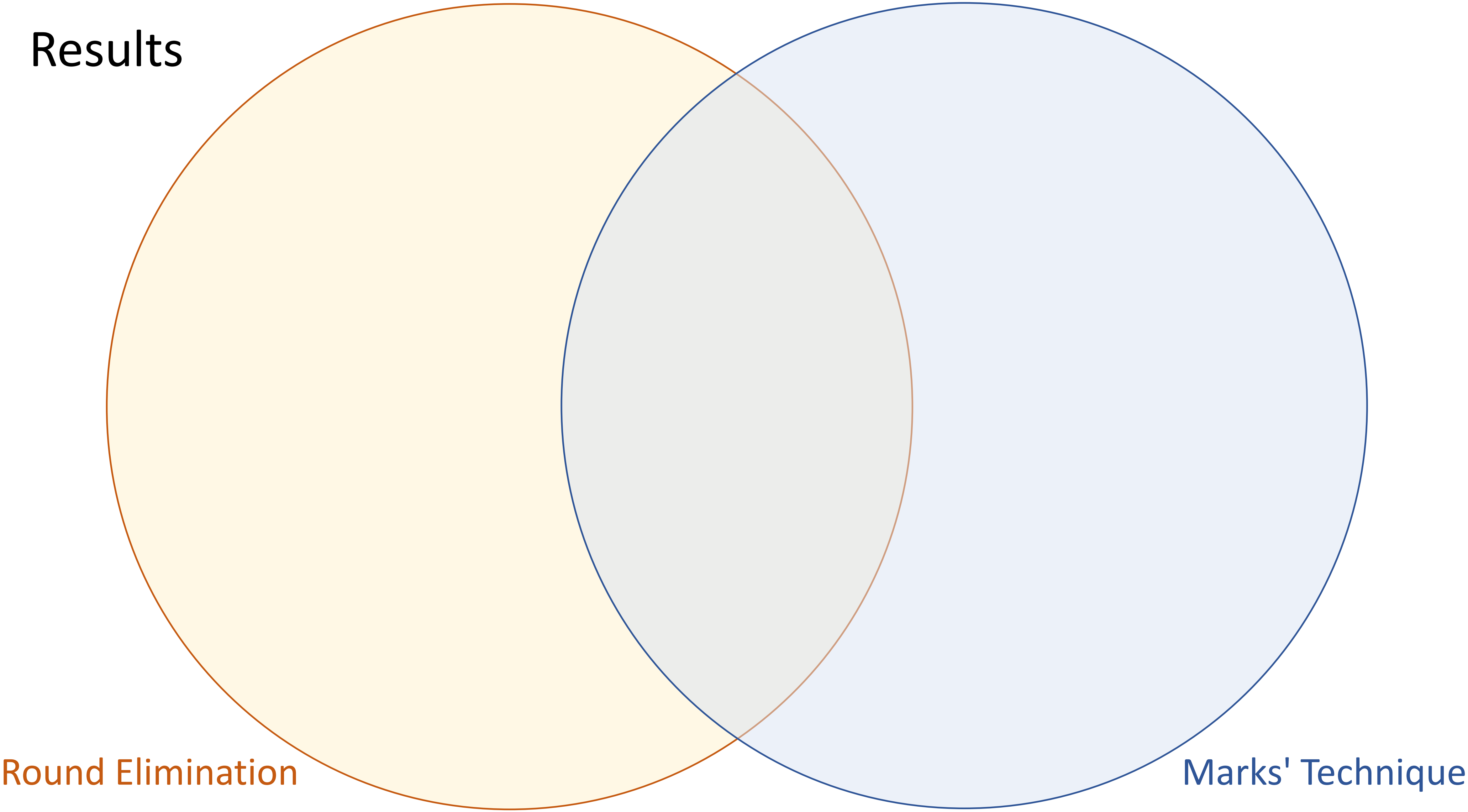
Round Elimination ←?→ Marks' Technique

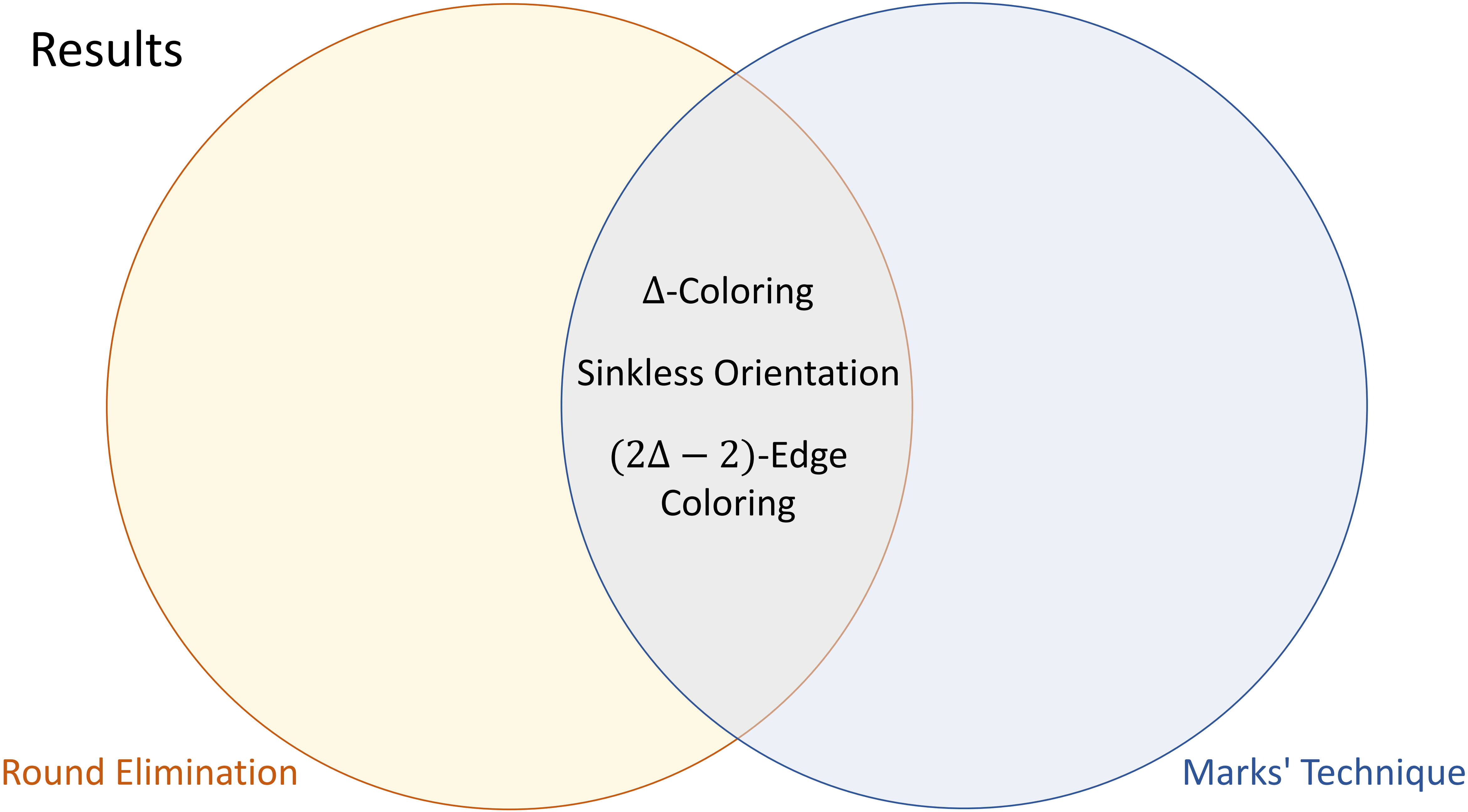


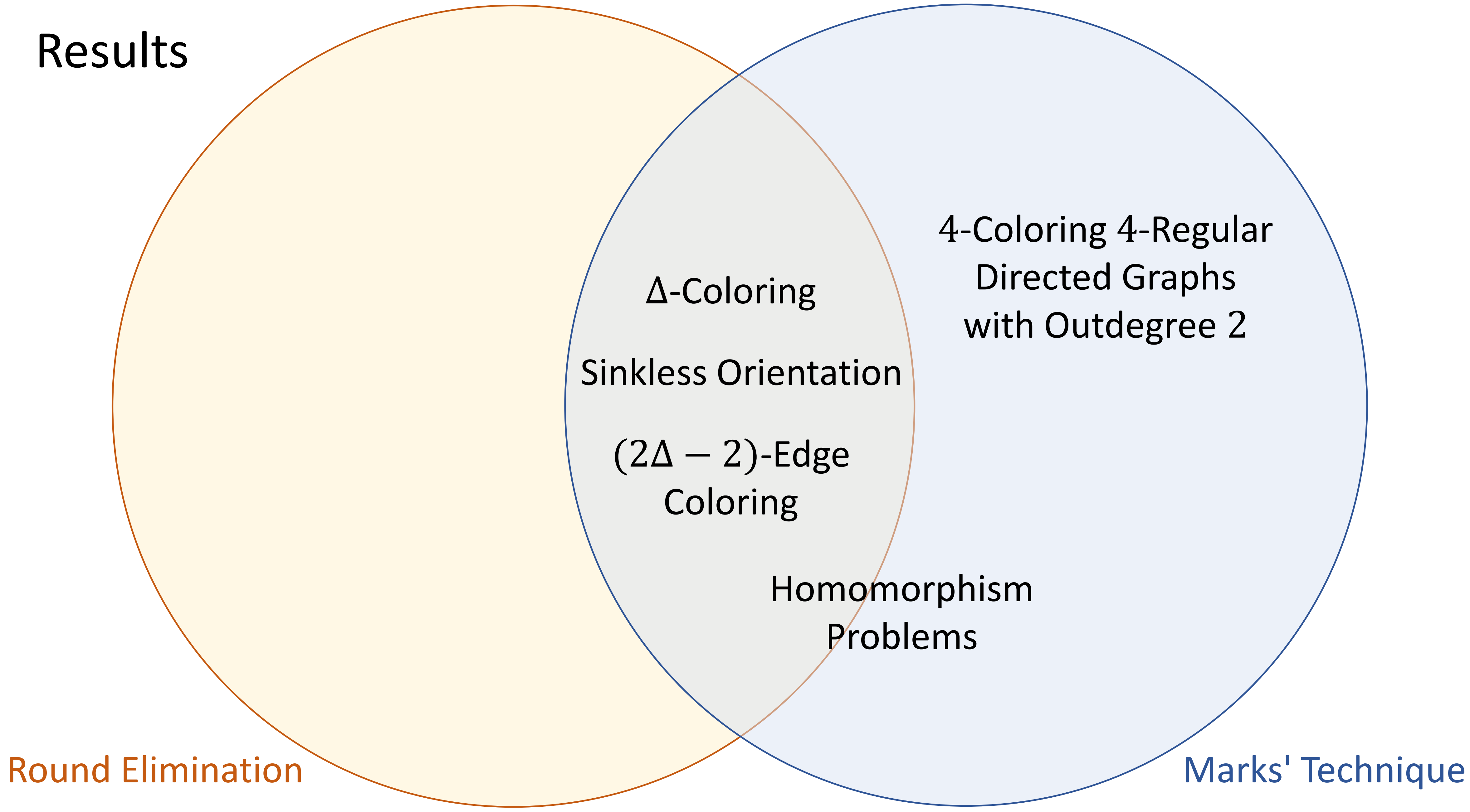


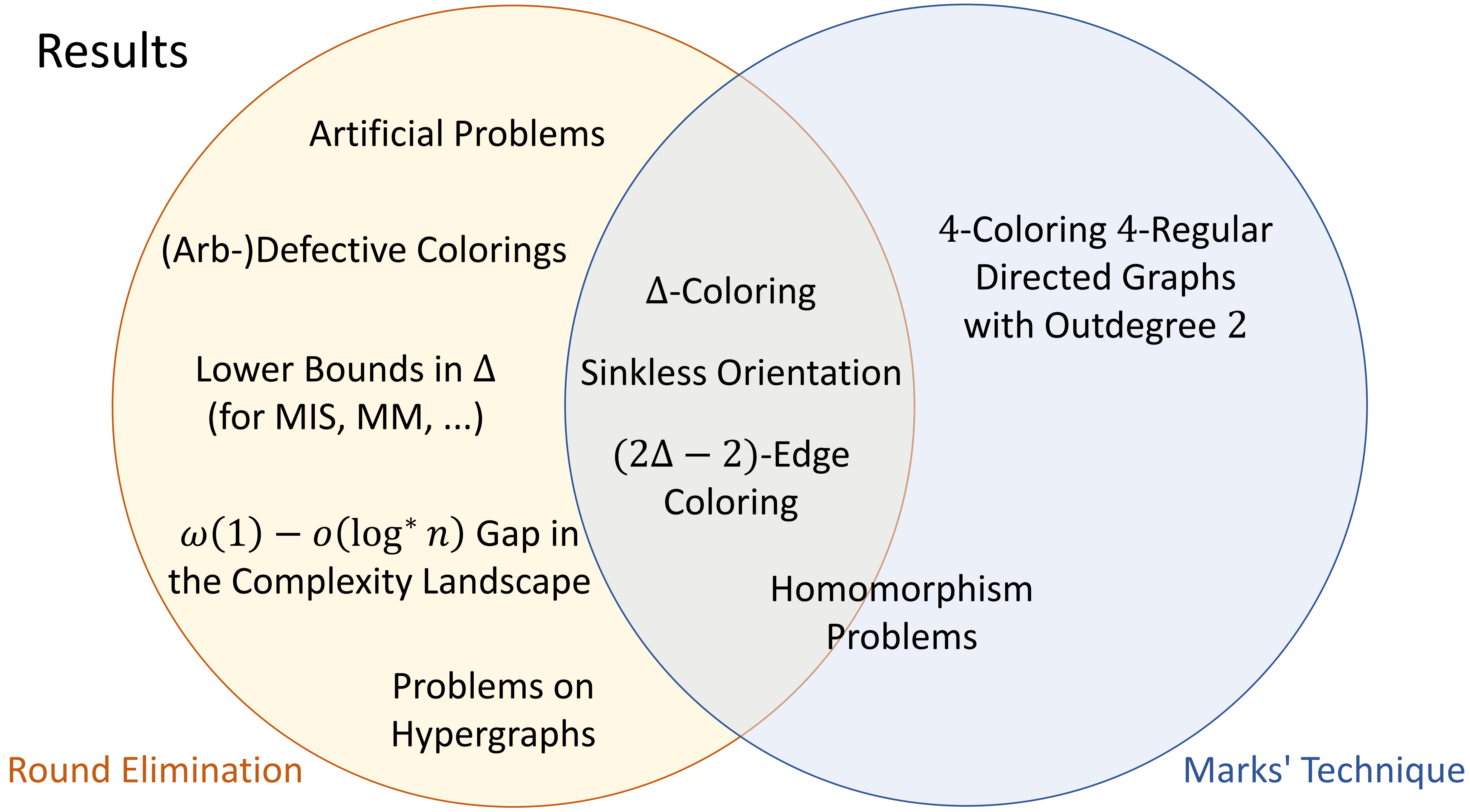
Results



Results


Δ-Coloring
Sinkless Orientation
(2Δ − 2)-Edge Coloring
Round Elimination
Marks' Technique


Results
Δ-Coloring
Sinkless Orientation
(2Δ − 2)-Edge Coloring
Homomorphism Problems
4-Coloring 4-Regular Directed Graphs with Outdegree 2
Round Elimination
Marks' Technique


Results
Artificial Problems
(Arb-)Defective Colorings
Lower Bounds in Δ
(for MIS, MM, ...)
$\omega(1) - o(\log^* n)$ Gap in
the Complexity Landscape
Problems on
Hypergraphs
Δ-Coloring
Sinkless Orientation
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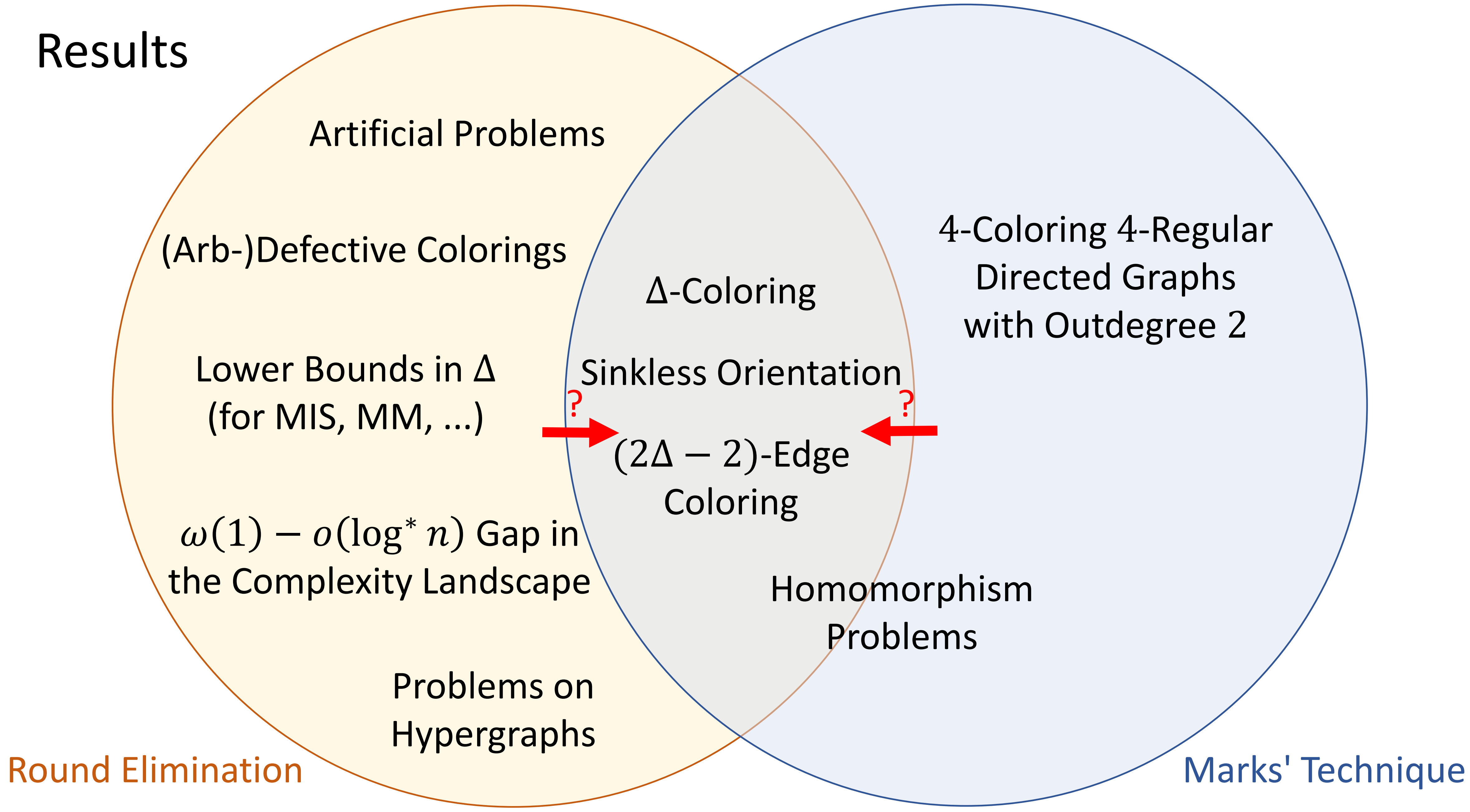
Round Elimination
Marks' Technique

Results



Artificial Problems
(Arb-)Defective Colorings
Lower Bounds in Δ
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$\omega(1) - o(\log^* n)$ Gap in
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Problems on
Hypergraphs
Δ-Coloring
Sinkless Orientation
?
?
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Coloring
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Marks' Technique

Open Problems

Open Problems

RE ⟶ Marks:

Randomized
Marks' Technique?

Bounds in Δ via
Marks' Technique?

Gaps, new RE, ... via
Marks' Technique?

Open Problems

RE $\longrightarrow$ Marks:

- Randomized Marks' Technique?
- Bounds in Δ via Marks' Technique?
- Gaps, new RE, ... via Marks' Technique?

Marks $\longrightarrow$ RE:

- Find good relaxations!
- Find fixed points!
- Find "properties" for round elimination!

Open Problems

RE ⟶ Marks:	Marks ⟶ RE:	Marks/RE:
Randomized Marks' Technique?	Find good relaxations!	Extend/Generalize Marks' Technique/Round Elimination!
Bounds in Δ via Marks' Technique?	Find fixed points!	Universality/Power of Marks' Technique/Round Elimination?
Gaps, new RE, ... via Marks' Technique?	Find "properties" for round elimination!	Marks' Technique/Round Elimination for other models?

Open Problems

RE ⟶ Marks:

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Bounds in Δ via Marks' Technique?

Gaps, new RE, ... via Marks' Technique?

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Find good relaxations!

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Extend/Generalize Marks' Technique/Round Elimination!

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Marks' Technique/Round Elimination for other models?

Marks & RE:

Invariance of applicability of Marks' Technique under Round Elimination?

Open Problems

RE ⟶ Marks:

Randomized Marks' Technique?

Bounds in Δ via Marks' Technique?

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Universality/Power of Marks' Technique/Round Elimination?

Marks' Technique/Round Elimination for other models?

Marks & RE:

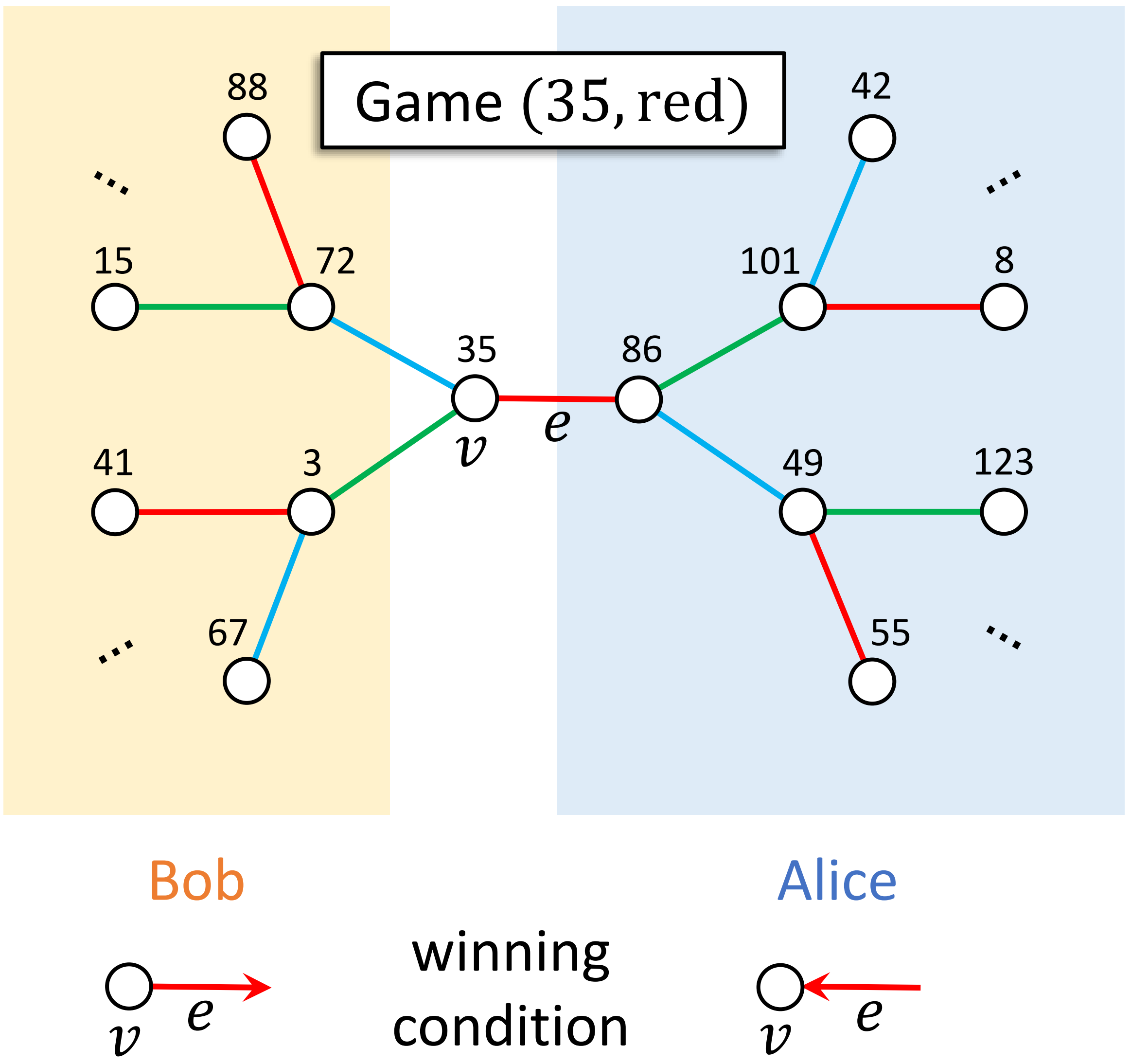
Invariance of applicability of Marks' Technique under Round Elimination?

Thanks!

Appendix: More Details about the Generalization of Marks' Technique

Marks' Technique

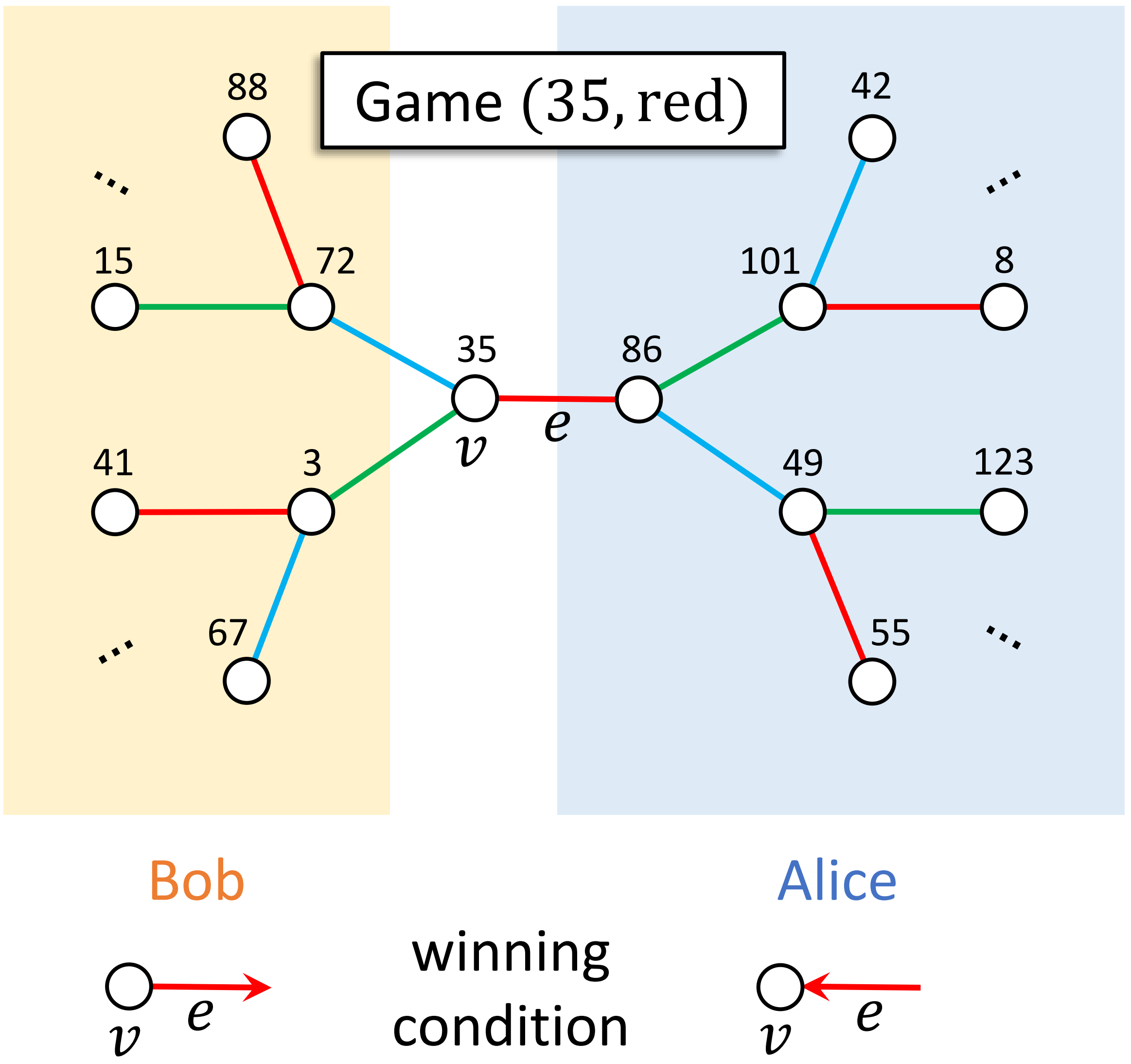
Sinkless Orientation



One game for each pair (ID, color).

Marks' Technique

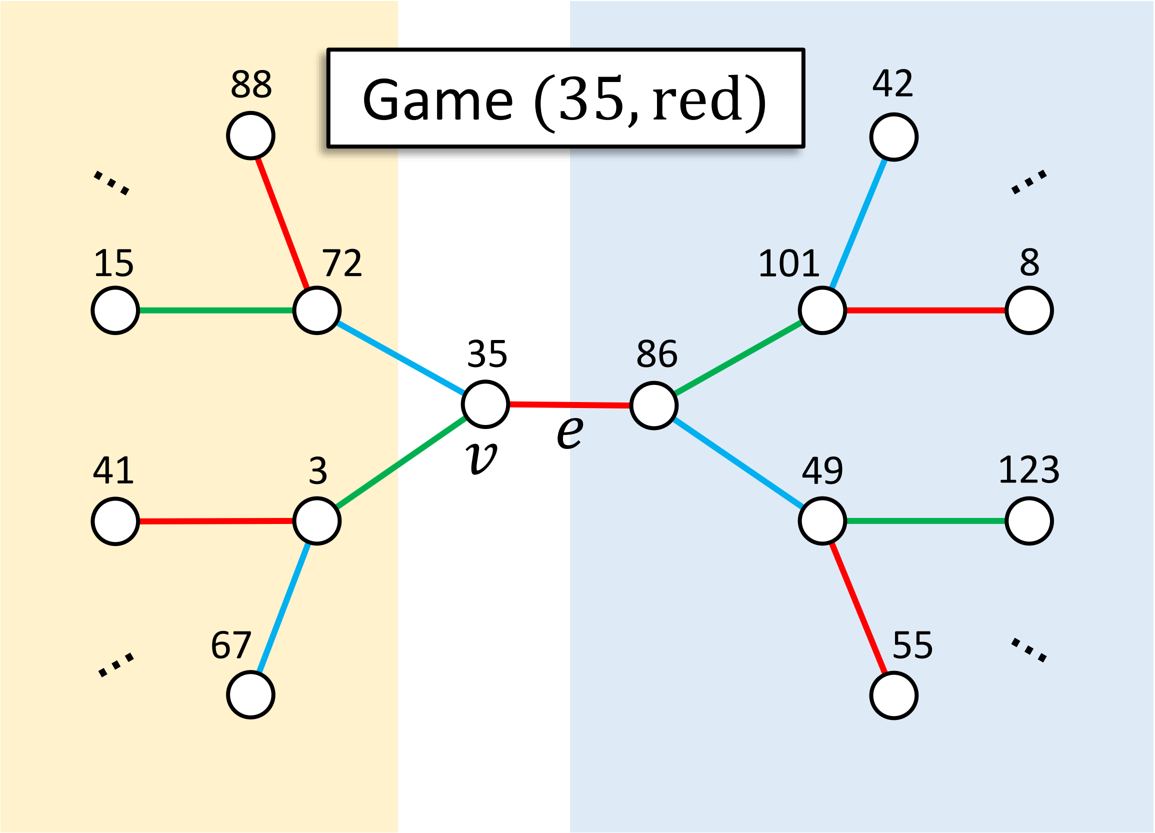
LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$



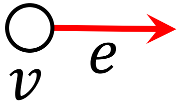
One game for each pair (ID, color).

Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$

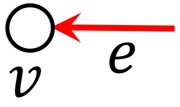


Bob



winning condition

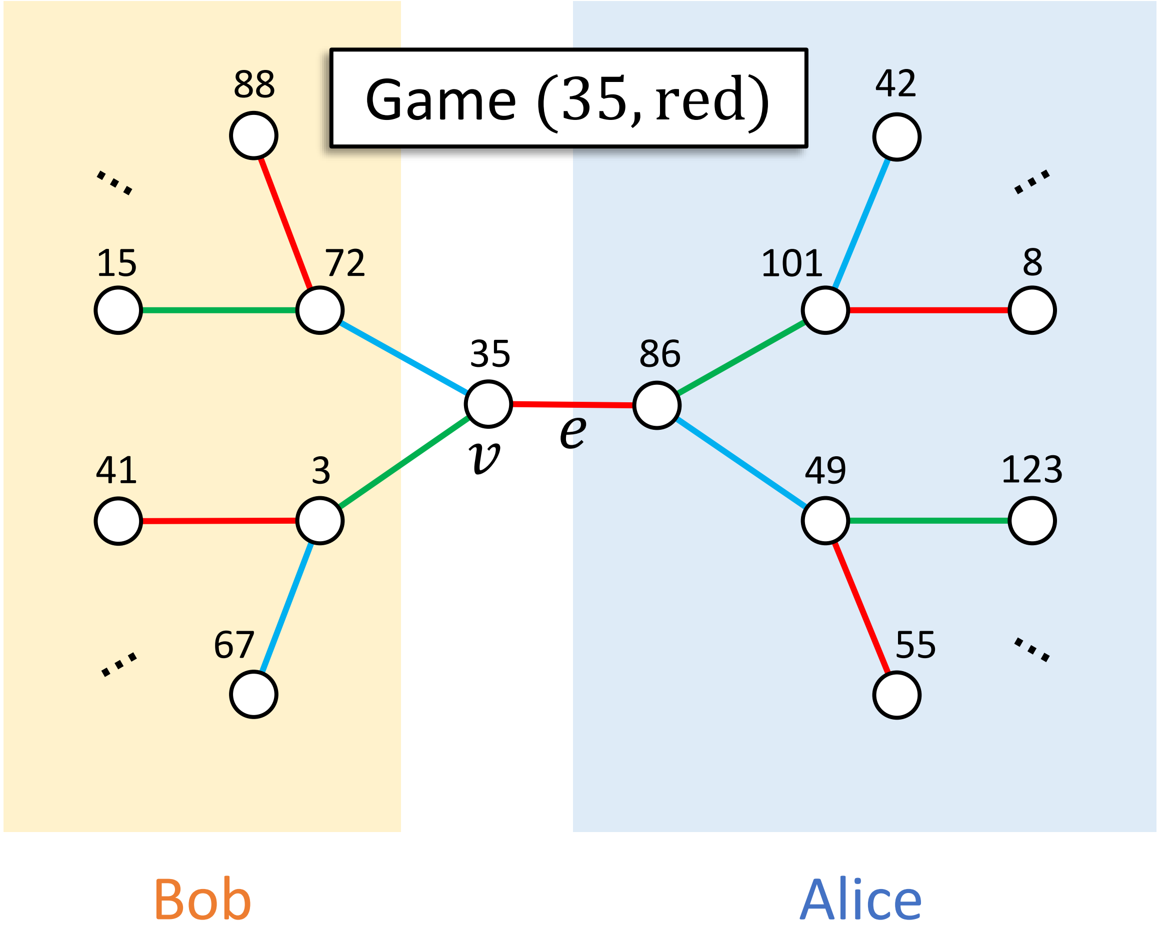
Alice

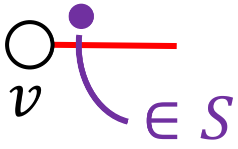


One game for each triple (ID, color, label subset S).

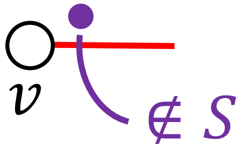
Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$





winning
condition



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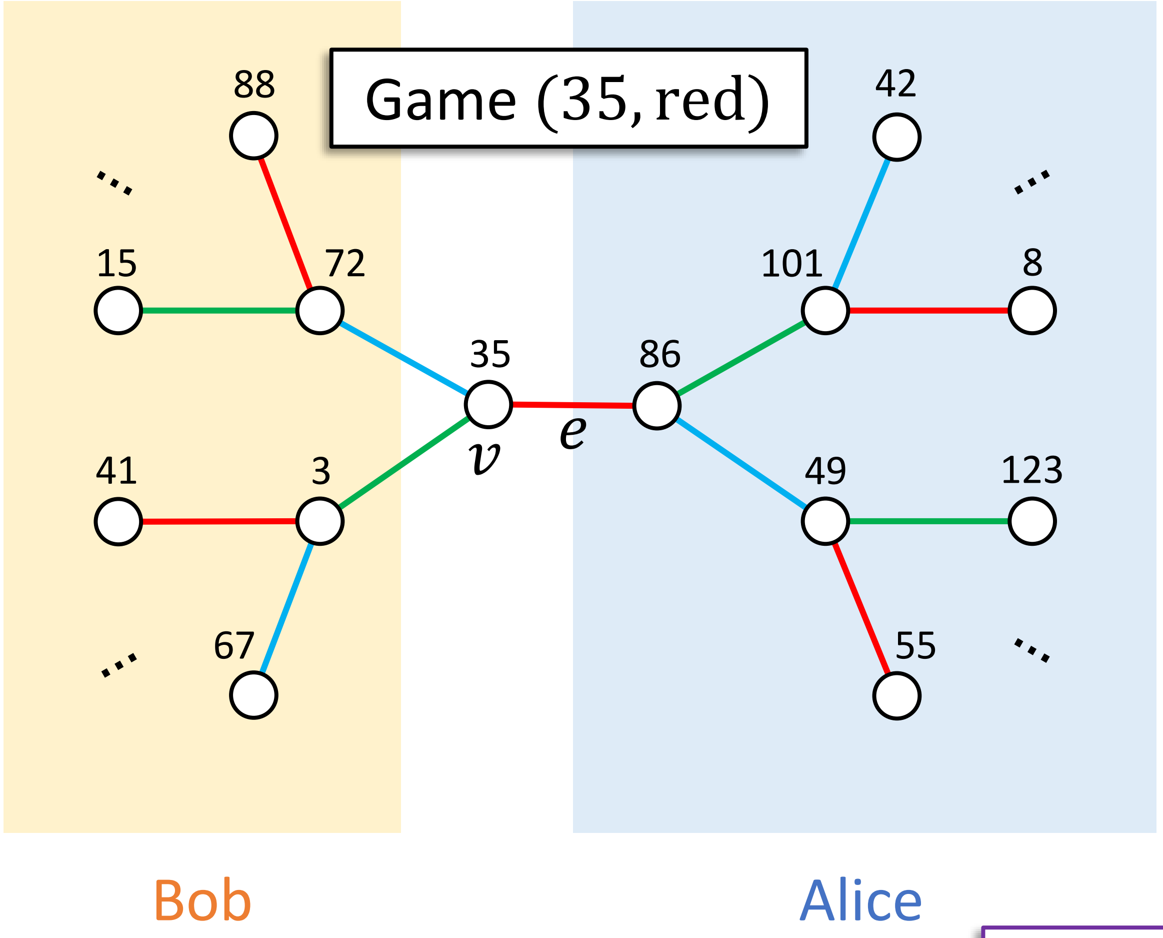
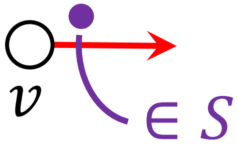
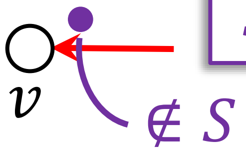
Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$





winning condition

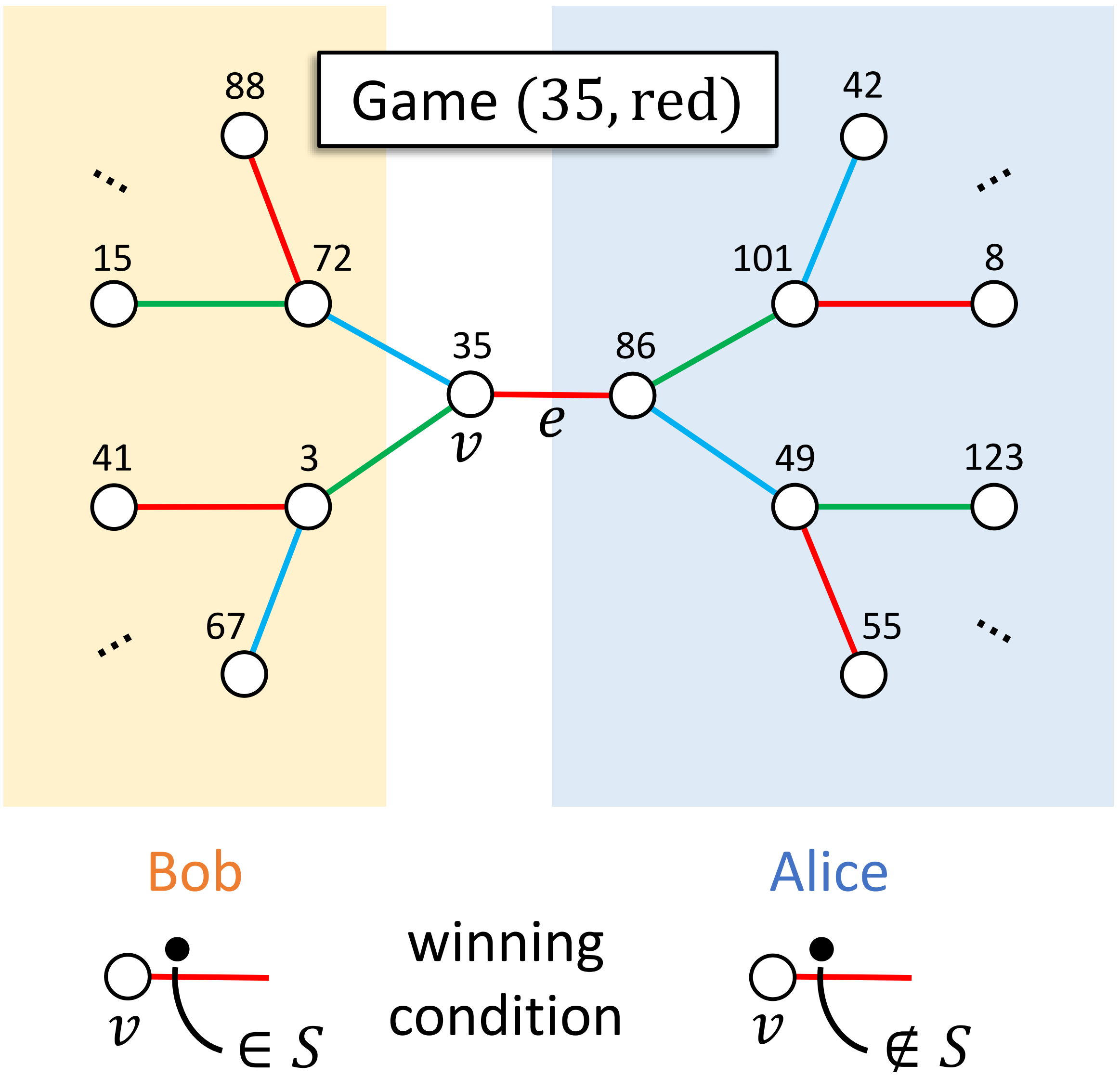


$S = \{O\}$

One game for each triple (ID, color, label subset S).

Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$



One game for each triple
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Marks' Technique

LCL problem
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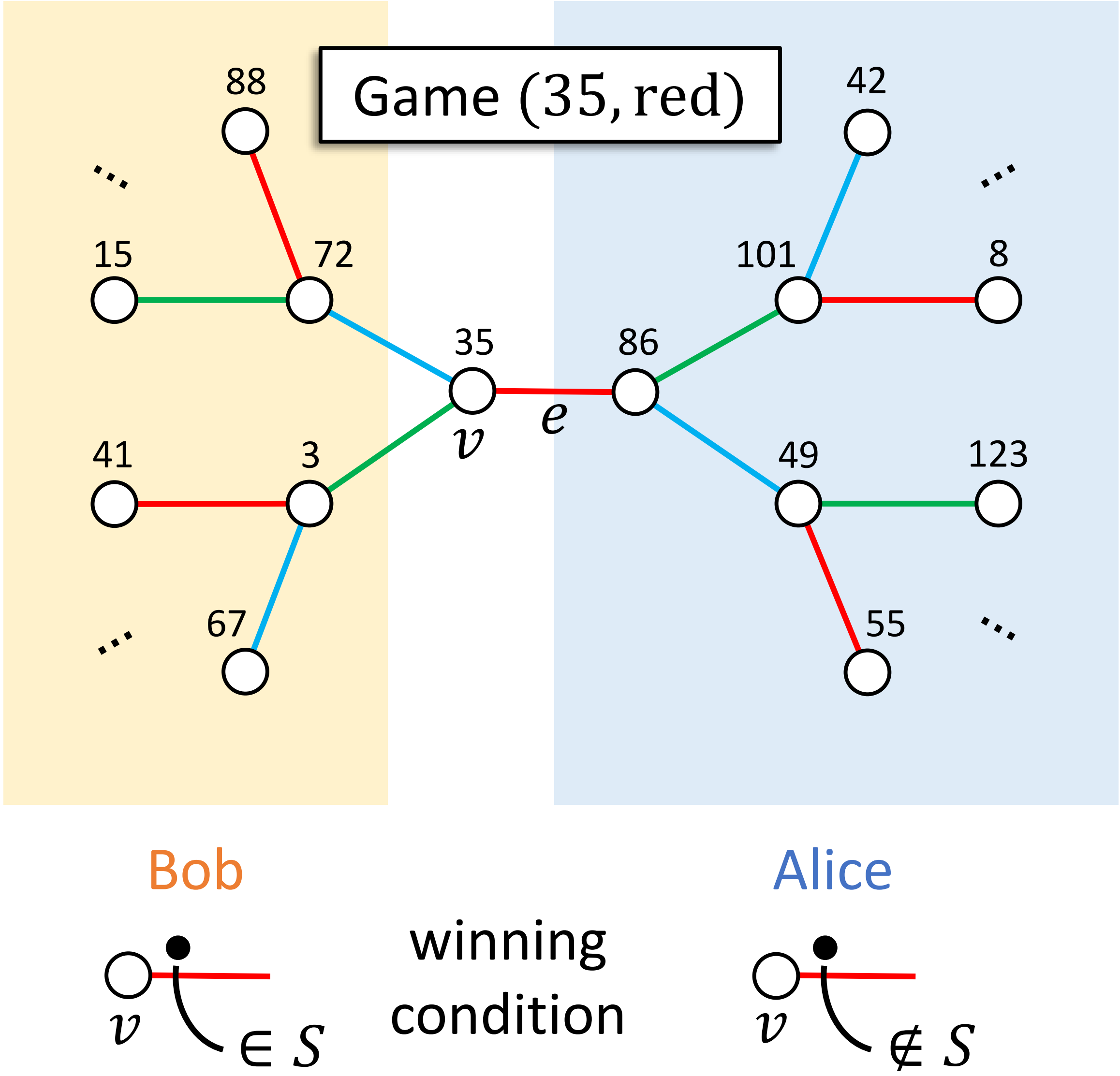
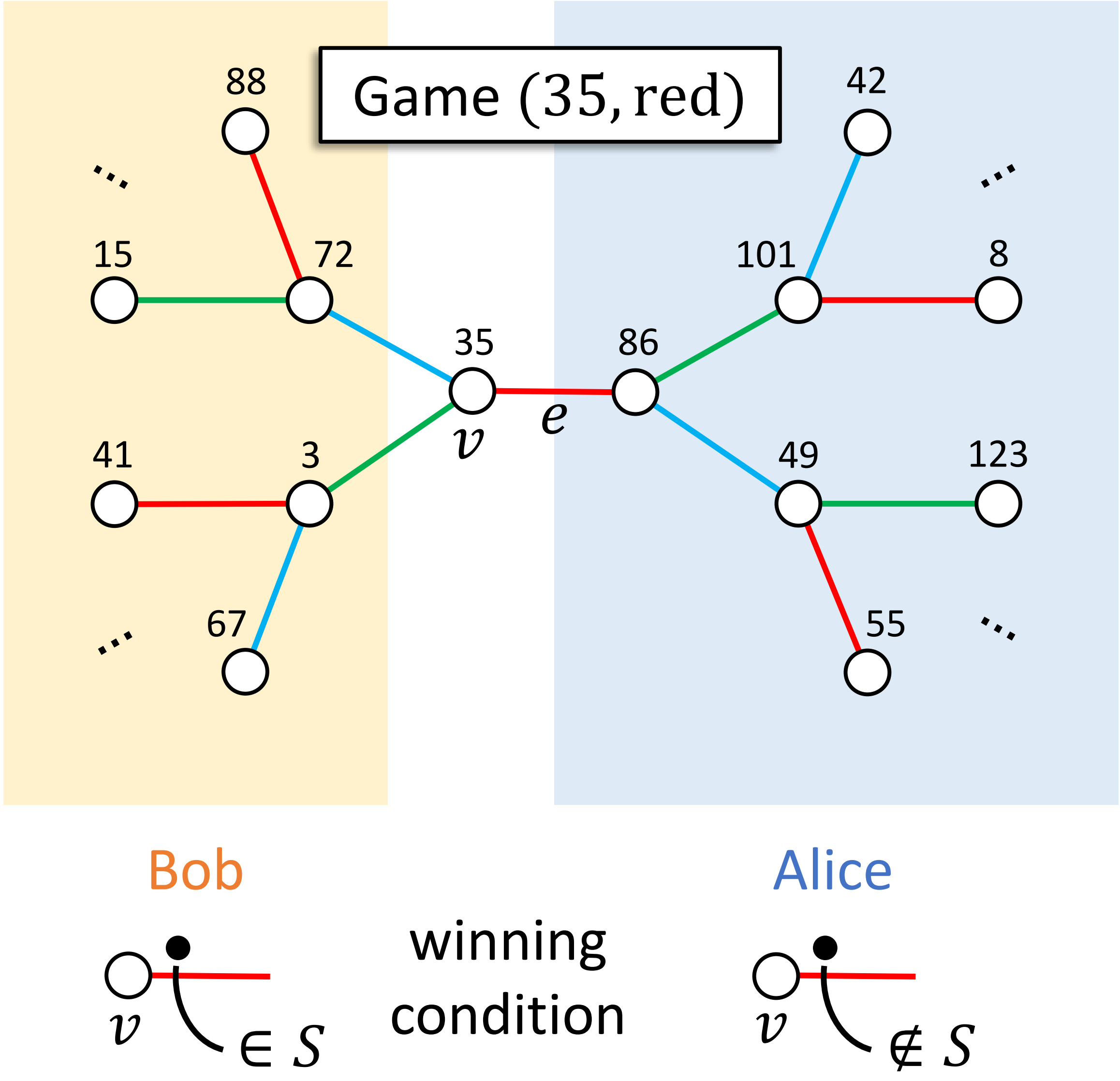


There are neighboring nodes in the ID graph that behave exactly the same w.r.t. who wins for which S.

One game for each pair (~~ID~~, color, label subset S).

Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$

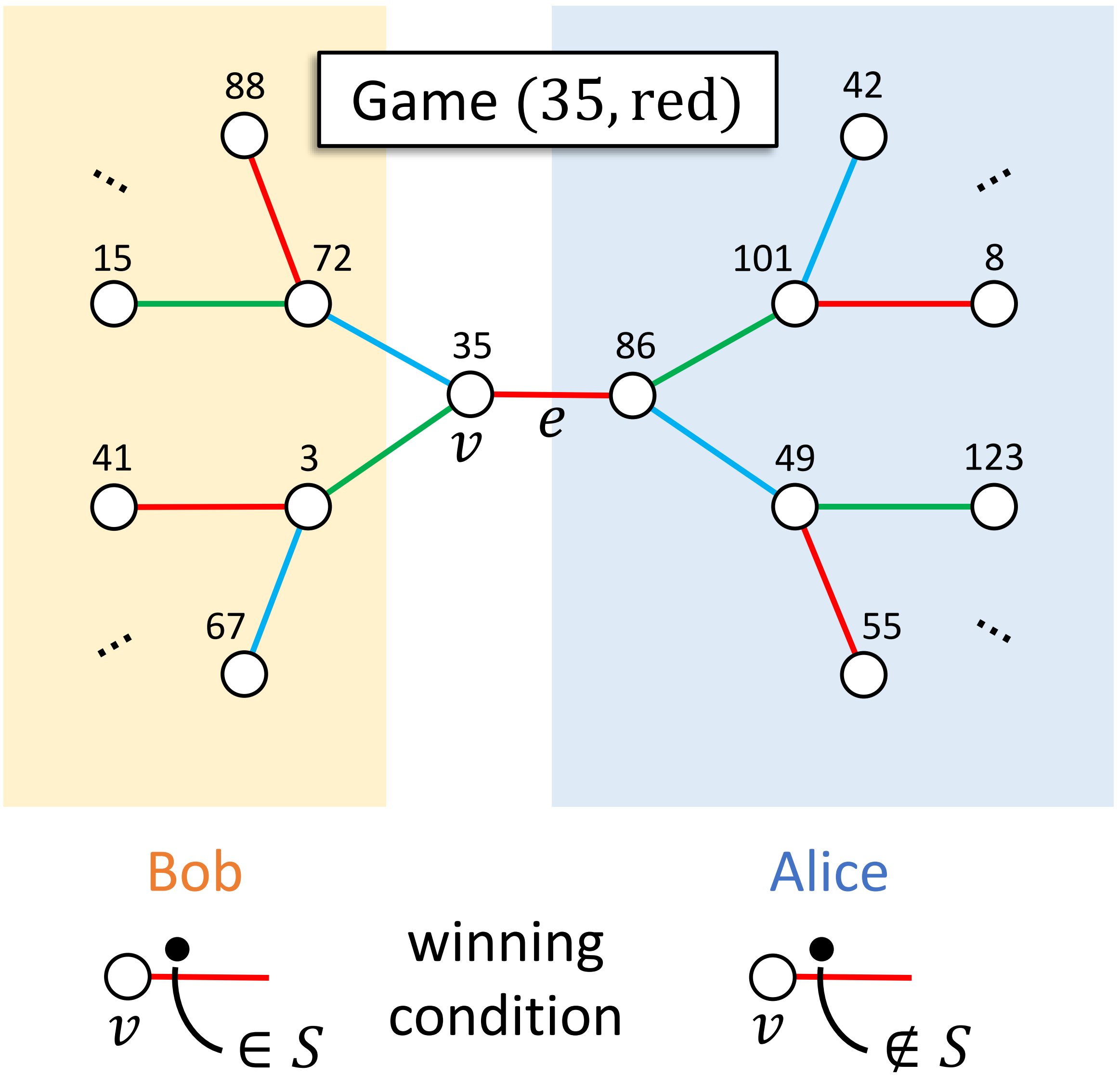


	red	green	blue
S_1			
S_2			
S_3			
S_4			
⋮			

One game for each pair
(~~ID~~, color, label subset S).

Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$



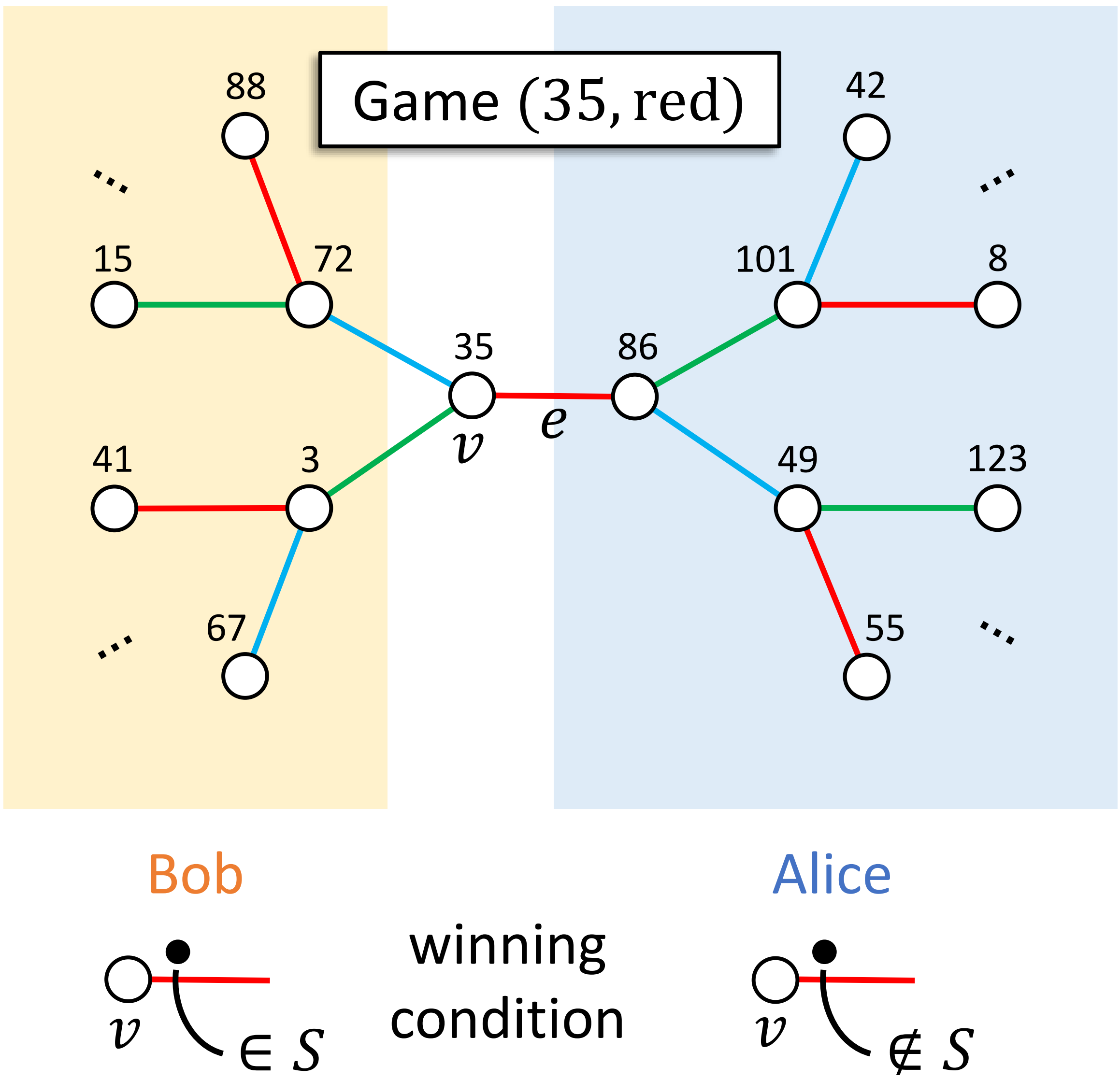
	red	green	blue
S_1			
S_2			
S_3			
S_4			
⋮			

Does there exist an assignment of winning strategies such that

- if Alice wins (red, S_i), (green, S_j), and (blue, S_k), then $\exists\ s_i \notin S_i, s_j \notin S_j, s_k \notin S_k$ such that $s_i\ s_j\ s_k \in \mathcal{N}$, and
- if Bob wins (color, S_i) and (color, S_j), then $\exists\ s_i \in S_i, s_j \in S_j$, such that $s_i\ s_j \in \mathcal{E}$?

Marks' Technique

LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$



	red	green	blue
S_1		A	
S_2			
S_3			A
S_4	A		
⋮			

Does there exist an assignment of winning strategies such that

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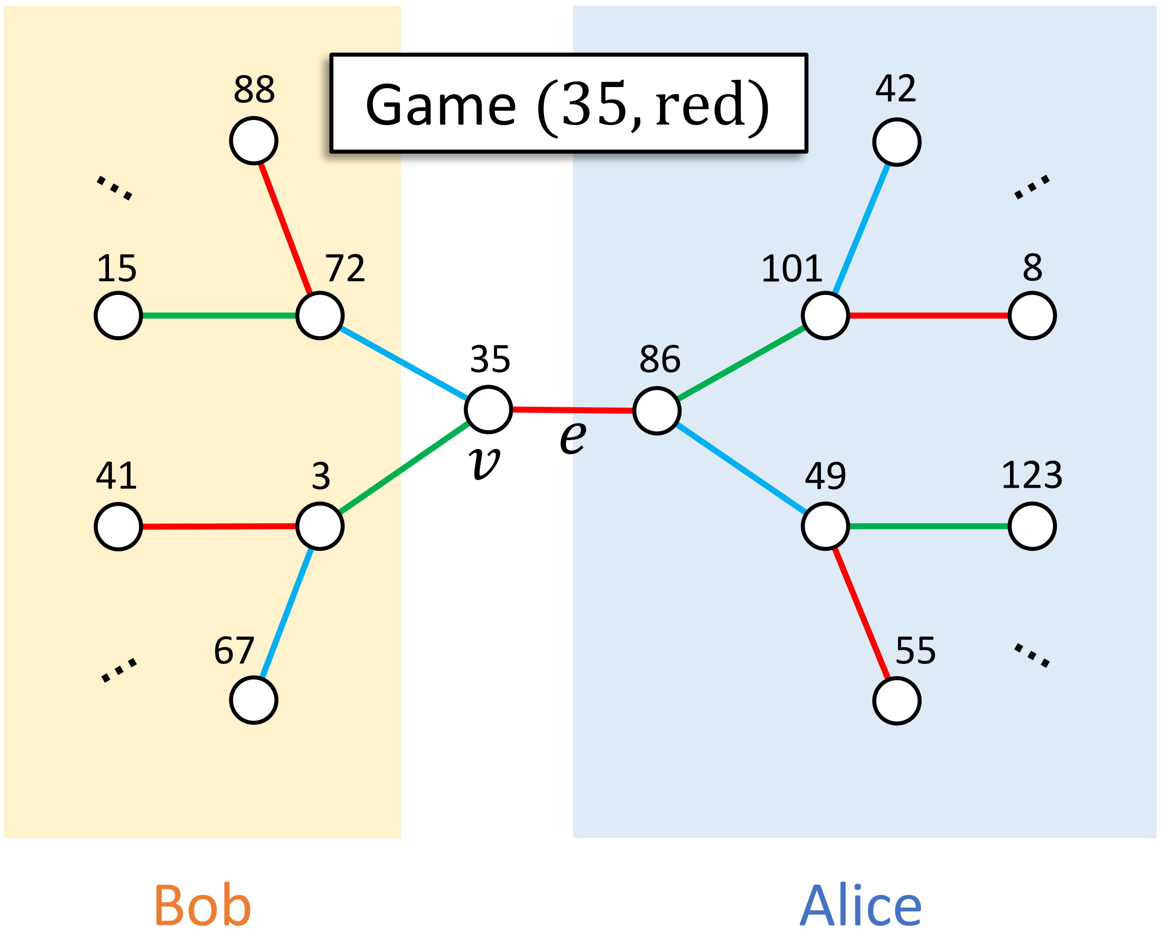
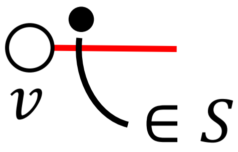
Marks' Technique

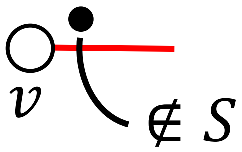
LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$

	red	green	blue
S_1		A	
S_2			B
S_3			A
S_4	A		B
⋮			





winning condition



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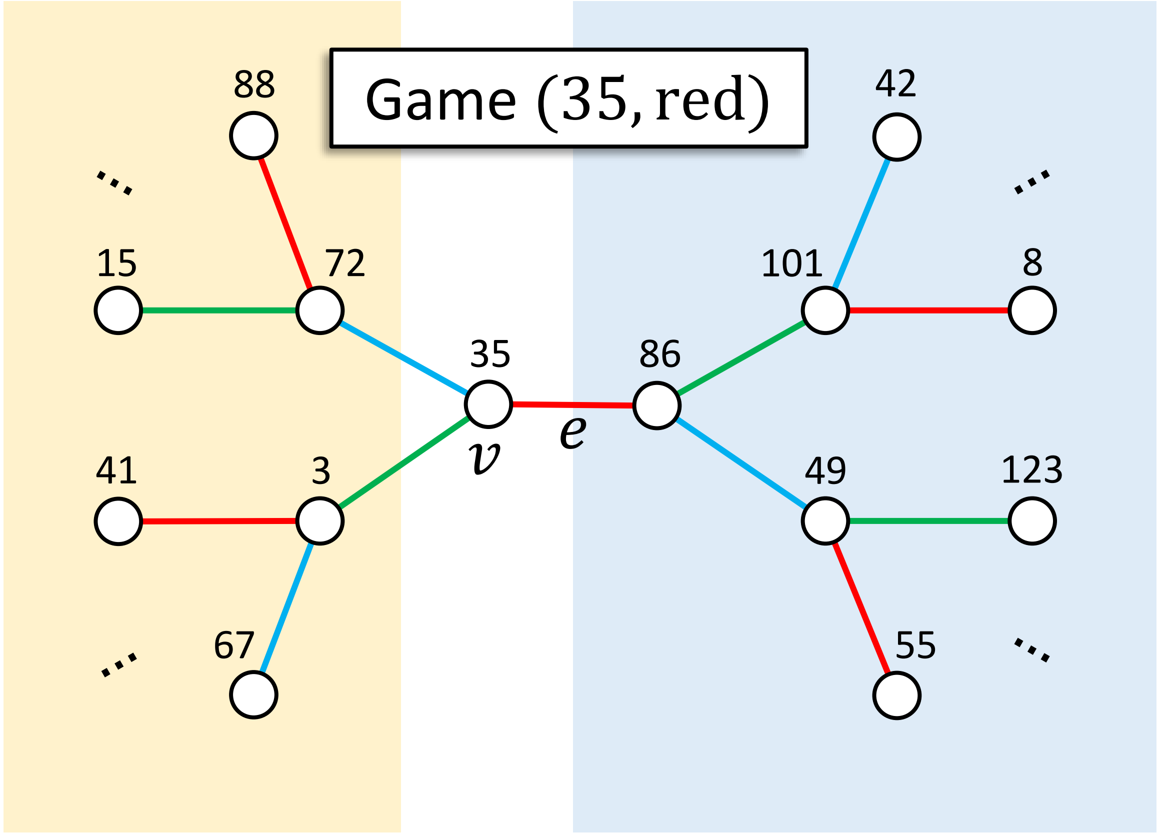
Marks' Technique

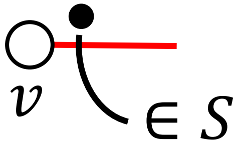
LCL problem
$\Pi = (\mathcal{N}, \mathcal{E})$

	red	green	blue
S_1		A	
S_2			B
S_3			A
S_4	A		B
⋮			



Bob

Alice



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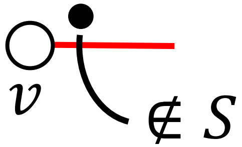

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No ⟶ $\Omega(\log n)$ lower bound