Hardness of Hinted ISIS from Space-Time Hardness of Lattice Problems

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Typical crypto theorems

$\nexists$ PPT algorithm for task A (e.g. solving discrete logarithm, finding short vectors in lattices)
$\implies$ $\nexists$ PPT algorithm for task B (e.g. breaking security of a scheme).

This work in a nutshell

$\nexists$ $2^{O(n)}$-time poly(n)-space probabilistic algorithm for finding short vectors in lattices
$\implies$ $\nexists$ PPT algorithm for Hinted Inhomogeneous Short Integer Solution (H-ISIS).

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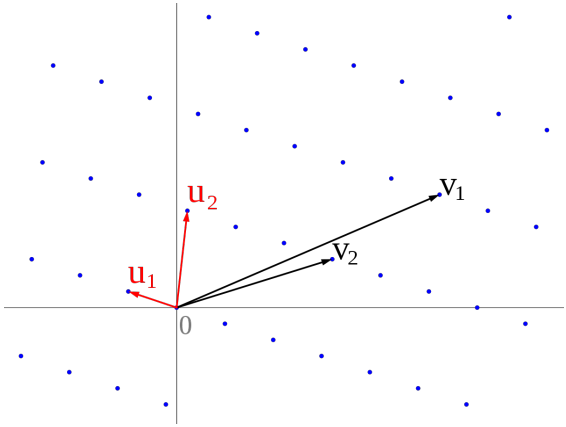
Lattices



$\mathbf{B} \in \mathbb{R}^{n \times k}$ basis, $k \leq n$

$$\mathcal{L}(\mathbf{B}) = \mathbf{B} \cdot \mathbb{Z}^k$$

"Full-rank" $\iff k = n$

q-ary lattices

- $\mathbb{Z}_q \coloneqq \mathbb{Z}/q\mathbb{Z}$
- $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ wide matrix, typically $2n \leq m \leq 2n \log q$
- $\Lambda_q^{\perp}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{Z}^m : \mathbf{A}\mathbf{x} = \mathbf{0} \bmod q\}$ "kernel lattice of $\mathbf{A}$"
- Note $q \cdot \mathbb{Z}^m \subseteq \Lambda_q^{\perp}(\mathbf{A}) \subseteq \mathbb{Z}^m$ are full-rank m-dimensional lattices.

Discrete Gaussians

$$\mathcal{D}_{\Lambda_q^\perp(\mathbf{A}),s,\mathbf{c}}$$

- Support restricted to $\Lambda_q^\perp(\mathbf{A})$
- Gaussian parameter s / standard deviation $\approx s/\sqrt{2\pi}$
- Centre $\mathbf{c}$, omitted if $\mathbf{c} = \mathbf{0}$.

Lattice problems

Approx. Shortest Independent Vector Problem ($\mathrm{SIVP}_{n,\alpha}$)

Let $\alpha = \alpha(n) \geq 1$ approximation factor. Given full-rank lattice $\Lambda \subset \mathbb{R}^n$, find linearly independent $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $\max\|\mathbf{v}_i\| \leq \alpha \cdot \lambda_n(\Lambda)$, where $\lambda_n(\Lambda)$ is the n-th successive minimum of Λ.

Short Integer Solution ($\mathrm{SIS}_{n,m,q,\beta}$)

Given $\mathbf{A} \leftarrow\$ \mathbb{Z}_q^{n\times m}$, find $\mathbf{v} \in \mathbb{Z}^m$ such that $\mathbf{A} \cdot \mathbf{v} \equiv \mathbf{0} \bmod q$ and $0 < \|\mathbf{v}\| \leq \beta$.

Inhomogeneous Short Integer Solution ($\mathrm{ISIS}_{n,m,q,\beta}$)

Given $\mathbf{A} \leftarrow\$ \mathbb{Z}_q^{n\times m}$ and $\mathbf{t} \leftarrow\$ \mathbb{Z}_q^n$, find $\mathbf{v} \in \mathbb{Z}^m$ such that $\mathbf{A} \cdot \mathbf{v} \equiv \mathbf{t} \bmod q$ and $\|\mathbf{v}\| \leq \beta$.

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Notation

(τ, μ)-SIS$_{n,m,q,\beta}$ (resp. (τ, μ)-ISIS$_{n,m,q,\beta}$ and (τ, μ)-SIVP$_{n,\alpha}$)

Set of (quantum) algorithms solving SIS$_{n,m,q,\beta}$ (resp. (τ, μ)-ISIS$_{n,m,q,\beta}$ and SIVP$_{n,\alpha}$) in time τ, memory μ with success probability at least $1/\mathrm{poly}(n)$.

E.g. "Standard" SIS assumption

For sensible n, m, q, β, it holds that $(\mathrm{poly}(n), \mathrm{poly}(n))$-SIS$_{n,m,q,\beta} = \emptyset$.

E.g. "Sub-exponential" SIS assumption

For sensible n, m, q, β, it holds that $(2^{o(n)}, 2^{o(n)})$-SIS$_{n,m,q,\beta} = \emptyset$.

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Reductions

SIVP $\implies$ SIS [Ajt96; MR07]

Parameters: $m, \beta \in \text{poly}(n)$, $q \geq 8\, n \cdot \sqrt{m} \cdot \beta$ and $\alpha \geq 8 \cdot \beta \cdot \sqrt{n} \cdot \omega(\sqrt{\log n})$.

$(\tau, \mu)\text{-SIS}_{n,m,q,\beta} \neq \emptyset \implies (\tau + \text{poly}(n), \mu + \text{poly}(n))\text{-SIVP}_{n,\alpha} \neq \emptyset$.

SIS $\implies$ ISIS

Parameters: $m, \beta \in \text{poly}(n)$, $\beta \geq \sqrt{n} \cdot \omega(\sqrt{\log n})$.

$(\tau, \mu)\text{-ISIS}_{n,m,q,\beta} \neq \emptyset \implies (\tau + \text{poly}(n), \mu + \text{poly}(n))\text{-SIS}_{n,m,q,\beta} \neq \emptyset$.

† [Ajt96]: Miklós Ajtai. "Generating Hard Instances of Lattice Problems (Extended Abstract)". In: *28th ACM STOC*. ACM Press, May 1996, pp. 99–108. DOI: 10.1145/237814.237838

† [MR07]: Daniele Micciancio and Oded Regev. "Worst-Case to Average-Case Reductions Based on Gaussian Measures". In: *SIAM Journal on Computing* 37.1 (2007), pp. 267–302. DOI: 10.1137/S0097539705447360. eprint: https://doi.org/10.1137/S0097539705447360

Algorithms for finding short vectors / solving SIS

Let $\Lambda \subset \mathbb{R}^n$ lattice. How to find a short vector in Λ (in worst-/average-case)?

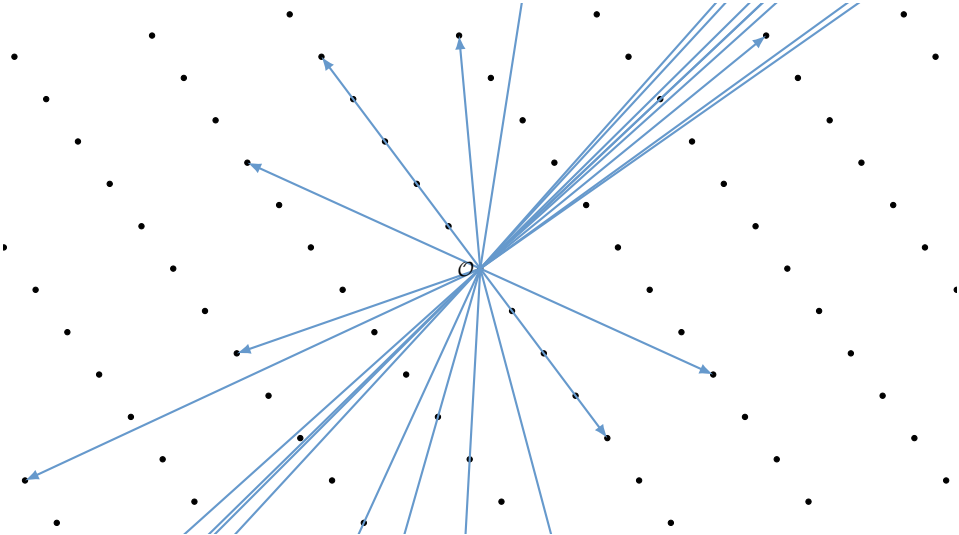
Enumeration

- Search through vectors smaller than a given bound: project down to 1-dim problem, lift to 2-dim problem ...
- **Time:** $n^{\Theta(n)}$
- **Memory:** $\mathrm{poly}(n)$

Sieving

- Produce new, shorter vectors by considering sums and differences of existing vectors
- **Time:** $2^{\Theta(n)}$
- **Memory:** $2^{\Theta(n)}$

Sieving



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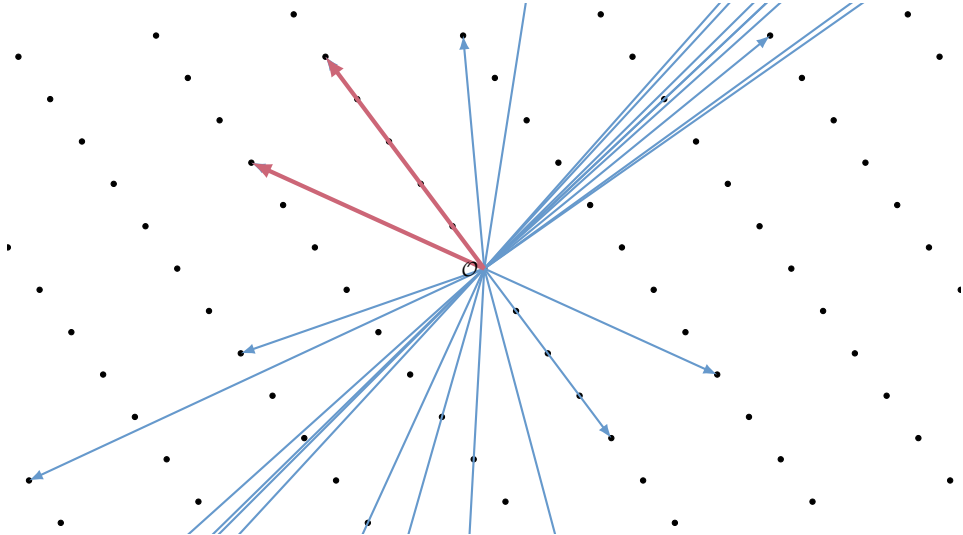

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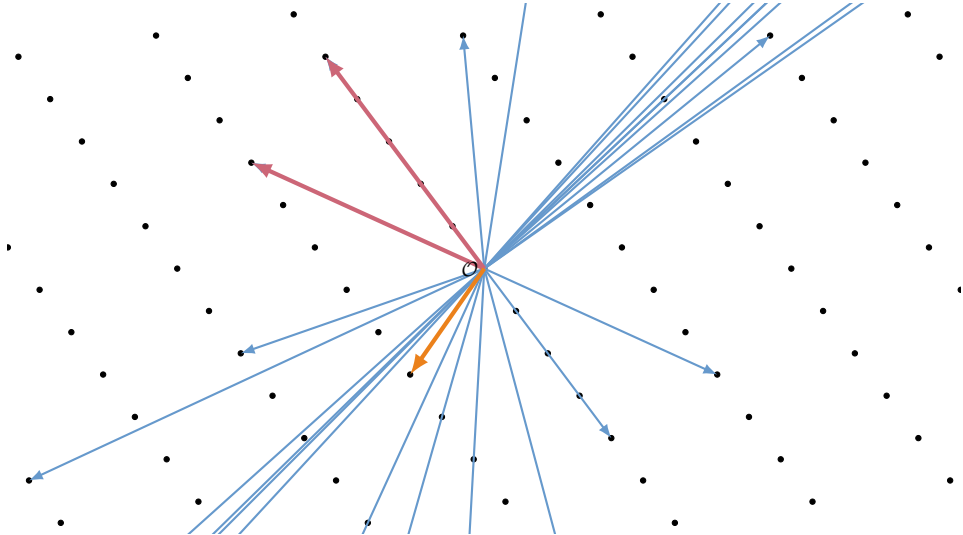

Space-time hardness: cryptanalysis

- ”It is still unclear if we can get a $2^{O(n)}$ algorithm that uses only polynomial space” [MV10]
- ”is it possible to achieve exponential time complexity with a polynomially bounded space requirement?” [HPS11]
- ”A more general open problem is whether SVP can be solved in singly exponential time but only polynomial space.” [ADRS15]

- [MV10]: Daniele Micciancio and Panagiotis Voulgaris. “Faster Exponential Time Algorithms for the Shortest Vector Problem”. In: *21st SODA*. ed. by Moses Charika. ACM-SIAM, Jan. 2010, pp. 1468–1480. DOI: 10.1137/1.9781611973075.119
- [HPS11]: Guillaume Hanrot, Xavier Pujol, and Damien Stehlé. “Algorithms for the Shortest and Closest Lattice Vector Problems”. In: *Coding and Cryptology - Third International Workshop, IWCC 2011*. Ed. by Yeow Meng Chee, Zhenbo Guo, San Ling, Fengjing Shao, Yuansheng Tang, Huaxiong Wang, and Chaoping Xing. Vol. 6639. Lecture Notes in Computer Science. Springer, 2011, pp. 159–190. DOI: 10.1007/978-3-642-20901-7_10
- [ADRS15]: Divesh Aggarwal, Daniel Dadush, Oded Regev, and Noah Stephens-Davidowitz. “Solving the Shortest Vector Problem in 2^n Time Using Discrete Gaussian Sampling: Extended Abstract”. In: *47th ACM STOC*. ed. by Rocco A. Servedio and Ronitt Rubinfeld. ACM Press, June 2015, pp. 733–742. DOI: 10.1145/2746539.2746606

Space-time hardness: conjecture

Corollary 2.2. *The conclusion of Theorem 2.2 follows from the assumption that worst-case* $\gamma(n)$-GapSVP *(for a fixed* $\gamma(n) = \mathrm{poly}(n)$*) cannot be solved in time* $n^{o(n)}$ *with* $\mathrm{poly}(n)$ *space and* $\mathrm{poly}(n)$ *bits of nonuniform advice (independent of the lattice).*

Polynomial-space algorithms for GapSVP have themselves been an object of study for over 25 years [Kan83, KF16, BLS16, ABF+20], but the current best (poly-space) algorithms for this problem run in time $n^{\Omega(\epsilon n)}$ for approximation factor $n^{1/\epsilon}$. Therefore, under a sufficiently strong (and plausible) worst-case assumption about GapSVP, we have a polynomial-time Fiat-Shamir compiler without complexity leveraging.

*Our $\alpha(n)$ is their $\gamma(n)$.

† [LV20]: Alex Lombardi and Vinod Vaikuntanathan. "Fiat-Shamir for Repeated Squaring with Applications to PPAD-Hardness and VDFs". In: *CRYPTO 2020, Part III*. ed. by Daniele Micciancio and Thomas Ristenpart. Vol. 12172. LNCS. Springer, Cham, Aug. 2020, pp. 632–651. DOI: 10.1007/978-3-030-56877-1_22

A corollary

From GapSVP $\leq$ SIVP $\leq$ SIS $\leq$ ISIS we get:

Assumption: Space-time hard ISIS

For sensible $m, q, \beta \in \text{poly}(n)$, $\left(2^{O(n)}, \text{poly}(n)\right)$ -ISIS$_{n,m,q,\beta} = \emptyset$.

A corollary (a specialisation)

Viewing $\mathrm{ISIS}_{n,m,q,\beta}$ as a lattice problem over $\Lambda_q^{\perp}(\mathbf{A})$, an m-dimensional lattice:

Assumption: Space-time hard ISIS

For sensible $q, \beta \in \mathrm{poly}(n)$, $m \in O(n)$ $\left(\mathbf{2^{O(m)}}, \mathrm{poly}(m)\right)$ -$\mathrm{ISIS}_{n,m,q,\beta} = \emptyset$.

- † When $m \leq O(n)$ then $2^{O(n)} = 2^{O(m)}$; we consider only this case in this talk.
- † In the paper we also consider $m \leq o(n \log n)$, which is covered by the conjecture.
- † When $m \geq \Omega(n \log n)$, there are algorithms in $\left(2^{O(m)}, \mathrm{poly}(m)\right)$ -$\mathrm{SIS}_{n,m,q,\beta}$.

Hinted ISIS (H-ISIS)

$\underline{\text{Exp-H-ISIS}_{(k,n,m,q,\beta,s),\mathcal{A}}(1^n)}$

$(\mathbf{A}, \mathbf{t}) \leftarrow\!\$\ \mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^n$

$\mathbf{u}_1, \ldots, \mathbf{u}_k \leftarrow\!\$\ \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}),s} \quad // \ \|\mathbf{u}_i\| \approx \sqrt{m} \cdot s$

$\mathbf{v} \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{t}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$

return $[\![\mathbf{A} \cdot \mathbf{v} \equiv \mathbf{t} \bmod q \wedge \|\mathbf{v}\| \leq \beta]\!]$

By default, we consider $m \ll k \in \text{poly}(n)$, i.e. many hints.

Parameter regimes

- $\beta \geq \Omega(m \cdot s)$: Trivially easy – run Babai's Nearest Plane algorithm
- $\beta \leq O(\sqrt{m} \cdot s)$: Plausibly hard
- $\beta = \sqrt{m} \cdot s$: Assumed for simplicity in this talk

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Main result

Theorem

$\left(2^{O(m)}, \mathrm{poly}(m)\right)$-ISIS$_{n,m,q,\tilde{\Omega}(m)\cdot\beta^*} = \emptyset$, where β^* is the optimal bound
$\implies$ There exists k, β, s such that $\left(\mathrm{poly}(n), \mathrm{poly}(n)\right)$-H-ISIS$_{k,n,m,q,\beta,s} = \emptyset$.

H-ISIS solver $\implies$ Malicious-centre Gaussian sampler

- Suppose $\mathcal{A}(\mathbf{A}, \mathbf{t}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ is an H-ISIS$_{k,n,m,q,\beta,s}$ solver (with probability 1 for simplicity).
- Recall $\mathbf{u}_1, \ldots, \mathbf{u}_k \hookleftarrow \$ \; \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}),s}$.
- Let $1 < \gamma \in O(1)$ "shrink factor" and $s' = s/\sqrt{2}\gamma$.

$\mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ // Essentially the standard SIS to ISIS reduction.

$\mathbf{x} \hookleftarrow \$ \; \mathcal{D}_{\mathbb{Z}^m,s'}$ // $\|\mathbf{x}\| \approx \sqrt{m} \cdot s'$

$\mathbf{t} := \mathbf{A} \cdot \mathbf{x} \bmod q$ // $(\mathbf{A}, \mathbf{t}) \approx (\mathbf{A}, \$)$

$\mathbf{c} \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{t}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$

return $(\mathbf{c}, \mathbf{c} - \mathbf{x})$

Interpretation: $\mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a statistically good sampler for the following distribution:

- First pick centre $\mathbf{c}$ from some (adversarial) distribution satisfying $\|\mathbf{c}\| \leq \sqrt{m} \cdot s$.
- Draw Gaussian sample $\mathbf{v} \hookleftarrow \$ \; \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}),s',\mathbf{c}}$.
  (This sample is quite long, $\|\mathbf{v}\| \approx \sqrt{m} \cdot s$, because $\mathbf{c}$ is quite long.)

Sieving the centres

$\mathcal{C}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$

1 : **for** $i = 1, \ldots, 2^{O(m)}$

2 : $\quad (\mathbf{c}'_i, \mathbf{v}'_i) \leftarrow \mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ / $\mathbf{v}'_i \leftarrow\!\$\ \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s', \mathbf{c}'_i}$

3 : $\quad (\mathbf{c}''_i, \mathbf{v}''_i) \leftarrow \mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$ / $\mathbf{v}''_i \leftarrow\!\$\ \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s', \mathbf{c}''_i}$

4 : $\quad$ **if** $\|\mathbf{c}'_i - \mathbf{c}''_i\| \leq \beta/(2\gamma)$

5 : $\quad\quad \mathbf{v}_i := \mathbf{v}'_i - \mathbf{v}''_i$ / $\mathbf{v}_i \sim \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), \sqrt{2}s', \mathbf{c}_i - \mathbf{c}'_i} = \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s/\gamma, \mathbf{c}_i - \mathbf{c}'_i}$

6 : $\quad\quad \mathbf{u}' \leftarrow \mathrm{RejectionSampling}(\mathbf{v}_i, \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s/\gamma})$ / A $1/2^{O(m)}$ fraction will be accepted.

7 : $\quad\quad$ **if** $\mathbf{u}' \neq \perp$

8 : $\quad\quad\quad$ **return** $\mathbf{u}'$

† If for some run Steps 4 and 7 both accept, $\mathcal{C}$ outputs a sample $\mathbf{u}' \leftarrow\!\$\ \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s/\gamma}$ – a factor γ improvement.

† Repeat to get a new set of hints $\{\mathbf{u}'_i\}_{1 \leq i \leq k}$ and feed back to $\mathcal{C}$.

† Remains to show that $\|\mathbf{c}'_i - \mathbf{c}''_i\| \leq \beta/(2\gamma)$ happens with probability at least $\frac{1}{2^{O(m)}}$.

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Geometry of sieves

Lemma: "There cannot be too many points in a ball that are far apart."

Let $S \subset \mathcal{B}^m(\beta)$.

If $|S| > 2^{m\log(1+4\gamma)}$, there exist distinct $\mathbf{c}', \mathbf{c}'' \in S$ with $\|\mathbf{c}' - \mathbf{c}''\| \leq \beta/2\gamma$.

Proof.

- Place balls of radii $r = \beta/4\gamma$ at points that are distance $\beta/2\gamma$ apart.
- Compare total volume of these balls with the volume of the ball with radius $\beta + r$.

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Proof.

- Place balls of radii $r = \beta/4\gamma$ at points that are distance $\beta/2\gamma$ apart.
- Compare total volume of these balls with the volume of the ball with radius $\beta + r$.

Probabilistic upgrade

Lemma: "No distribution over a ball can avoid close pairs."

For **any (malicious) distribution** D over $\mathcal{B}^m(\beta)$ with finite support:

$$\Pr_{\mathbf{c}',\mathbf{c}''\leftarrow D}\left[\left\|\mathbf{c}'-\mathbf{c}''\right\| \le \beta/2\gamma\right] \ge 2^{-m\log(1+4\gamma)}. \quad \text{/ Note: } \mathbf{c}'=\mathbf{c}'' \text{ falls into the event.}$$

Motzkin–Straus Theorem [MS65]

Let G undirected simple graph, clique number $\omega(G)$, and adjacency matrix $\mathbf{M}$.

$$\max_{\mathbf{d}\in[0,1]^n:\|\mathbf{d}\|_1=1} \mathbf{d}^T\cdot\mathbf{M}\cdot\mathbf{d} = 1-\frac{1}{\omega(G)}.$$

Proof of Lemma. Define G so that $\mathbf{x}$ and $\mathbf{y}$ are connected iff $\|\mathbf{x}-\mathbf{y}\| > \beta/2\gamma$. Clique = "(anti-)cluster" of far-apart points. Apply Motzkin–Straus on G. By geometry of sieves, $\omega(G) \le 2^{m\log(1+4\gamma)}$.

† [MS65]: T. S. Motzkin and E. G. Straus. "Maxima for Graphs and a New Proof of a Theorem of Turán". In: *Canadian Journal of Mathematics* 17 (1965), pp. 533–540. DOI: 10.4153/CJM-1965-053-6

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Starting and stopping conditions

$\mathcal{C}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$

1 : **for** $i = 1, \ldots, 2^{O(m)}$
2 : $\quad (\mathbf{c}_i', \mathbf{v}_i') \leftarrow \mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$
3 : $\quad (\mathbf{c}_i'', \mathbf{v}_i'') \leftarrow \mathcal{B}(\mathbf{A}, \mathbf{u}_1, \ldots, \mathbf{u}_k)$
4 : $\quad$ **if** $\|\mathbf{c}_i' - \mathbf{c}_i''\| \leq \beta/(2\gamma)$
5 : $\quad\quad \mathbf{v}_i := \mathbf{v}_i' - \mathbf{v}_i''$
6 : $\quad\quad \mathbf{u}' \leftarrow \mathsf{RejSamp}(\mathbf{v}_i, \mathcal{D}_{\Lambda_q^\perp(\mathbf{A}), s/\gamma})$
7 : $\quad\quad$ **if** $\mathbf{u}' \neq \perp$
8 : $\quad\quad\quad$ **return** $\mathbf{u}'$

Starting condition

We kickstart $\mathcal{C}$ with trivial hints, $\|\mathbf{u}_i\| \approx \sqrt{m} \cdot q$.

Stopping conditions

Norm bound $\beta \geq \Omega(m) \cdot \lambda_1(\Lambda_q^\perp(\mathbf{A}))$ violated.

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Caveats

† We need to assume existence of a chain of adversaries for different parameters.

† The whole chain of adversaries need to be happy with the same **A**.

$\Longrightarrow$ (Conditionally) rule out H-ISIS adversaries for **some** parameters but we don't know which.

Take away

Theorem

$\left(2^{O(m)}, \mathrm{poly}(m)\right)$-$\mathrm{ISIS}_{n,m,q,\tilde{\Omega}(m)\cdot\beta^*} = \emptyset$, where β^* is the optimal bound
$\implies$ There exists k, β, s such that $\left(\mathrm{poly}(n), \mathrm{poly}(n)\right)$-$\mathrm{H\text{-}ISIS}_{k,n,m,q,\beta,s} = \emptyset$.

Directions

1. What can we build from H-ISIS?
2. Reduce space-time hardness of lattice problems to other hinted lattice problems?
   In particular, Hinted **Decision** Learning with Errors?

† https://ia.cr/2026/187

† Crypto Seminar every other Wed 11-12

† More info: https://research.cs.aalto.fi/crypto/

Kiitos!